AIM Preliminary Exam: Differential Equations & Linear Algebra

January 8, 2011

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

Consider the system of differential equations

$$\frac{dx}{dt} = 2x - x^2 - xy, \quad \frac{dy}{dt} = 2y - y^2 - xy.$$

(a) Note that this system preserves the axes in the coordinate plane. Find the exact solutions corresponding to the two coordinate axes.

(b) Exploit an obvious symmetry of the system to find an additional family of exact solutions.

(c) Find all fixed points of the system in the quarter-plane $x \geq 0$, $y \geq 0$.

(d) Determine the type of each of the fixed points from part (c) by linearization. What can you deduce about the behavior of the nonlinear system, and why?

(e) Sketch a careful phase portrait of the system in the quarter-plane $x \geq 0$, $y \geq 0$.

(f) Let $x(t; x_0, y_0)$ and $y(t; x_0, y_0)$ denote the solution of the nonlinear system with initial conditions $x(0; x_0, y_0) = x_0$ and $y(0; x_0, y_0) = y_0$. Define

$$X(x_0, y_0) := \lim_{t\to\infty} x(t; x_0, y_0) \quad \text{and} \quad Y(x_0, y_0) := \lim_{t\to\infty} y(t; x_0, y_0),$$

whenever the limits exist. Calculate (X, Y) as a function of $x_0 \geq 0$ and $y_0 \geq 0$.

Problem 1

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Problem 2

Consider the $N \times N$ $(N > 1)$ matrix $\mathbf{A}$ with complex entries, of the form

$$\mathbf{A} := \begin{bmatrix} 2s & t & t & \cdots & t \\ t & 2s & t & \cdots & t \\ t & t & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 2s & t \\ t & t & \cdots & t & 2s \end{bmatrix}$$

in which all diagonal entries are $2s$ with $s \in \mathbb{C}$ and all off-diagonal entries are $t \in \mathbb{C}$. Determine all values $(s, t) \in \mathbb{C}^2$ for which $\mathbf{A}$ is invertible.

Problem 2

Problem 2

Problem 2

Problem 3

Consider the equation $\mathbf{A}^3 - 2\mathbf{A} + \mathbb{I} = \mathbf{0}$, where $\mathbf{A}$ is a 3×3 matrix .

(a) Find the general diagonalizable solution.

(b) Now find the (completely) general solution. Hint: Jordan form.

Problem 3

Problem 3

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Problem 4

(a) Suppose $\mathbf{A}$ is a real $N \times N$ matrix. Prove that $\mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x} = 0$ for all $\mathbf{x} \in \mathbb{R}^N$ if and only if $\mathbf{A}^\mathsf{T} = -\mathbf{A}$. Hint: recall the spectral theorem.

(b) Suppose that $\mathbf{A}$ is a complex $N \times N$ matrix. Prove that $\mathbf{x}^\dagger\mathbf{A}\mathbf{x} = 0$ for all $\mathbf{x} \in \mathbb{C}^N$ if and only if $\mathbf{A} = \mathbf{0}$.

Problem 4

Problem 4

Problem 4

Problem 5

Consider the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{1}{64}\cos(t/8)$$

in the domain $x \in [0, \pi]$. The boundary conditions are:

$$u(0,t) = 1 - \cos(t/8) \quad \text{and} \quad \frac{\partial u}{\partial x}(\pi, t) = 0.$$

The initial conditions are

$$u(x,0) = \sin(x/2) + \sin(3x/2) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = 0.$$

Find $u(x,t)$.

Problem 5

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