AIM Qualifying Review Exam in Differential Equations & Linear Algebra

September 3, 2016

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

Let the $\mathbf{A}$ be a 2-by-2 matrix with real entries, $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The range of $\mathbf{A}$ is $\{\mathbf{A}\mathbf{x} : \mathbf{x} \in \mathbb{R}^2\}$.

(a) (8 points) List all possible values of the rank (the dimension of the range) of $\mathbf{A}$. For each value of the rank, find necessary and sufficient conditions on a, b, c, and d such that $\mathbf{A}$ has that rank.

(b) (12 points) Now for each value of the rank, find necessary and sufficient conditions on a, b, c, and d such that $\mathbf{A}$ has that rank *and* $\mathbf{A}$ *is a projection matrix*, i.e. $\forall \mathbf{v} \in$ range of $\mathbf{A}$, $\mathbf{A}\mathbf{v} = \mathbf{v}$.

Problem 1

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Problem 2

(a) (10 points) Compute the determinant of $\mathbf{B} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & 2 & 3 & 4 \\ 0 & 4 & 1 & 2 \\ 0 & 0 & 5 & 4 \end{pmatrix}$.

(b) (10 points) Find the solution to $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ which lies closest to the origin.

Problem 2

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Problem 3

(a) (10 points) Find the eigenvalues and eigenfunctions of

$$y'' - 4\lambda y' + 4\lambda^2 y = 0 \tag{1}$$

with the boundary conditions

$$y(0) = 0, \quad y(1) + y'(1) = 0. \tag{2}$$

In other words, find all solutions $y(x)$ other than the zero function, and corresponding values of the constant λ.

(b) (10 points) Find a linear scalar ODE with constant coefficients of the least possible order which has $x_1(t) = 1$ and $x_2(t) = \cos t$ as particular solutions.

Problem 3

Problem 3

Problem 3

Problem 4

The functions t, t^5, and $|t|^5$ are solutions to the differential equation $t^2x'' - 5tx' + 5x = 0$.

(a) (5 points) Are the solutions linearly independent on $-1 < t < 1$?

(b) (15 points) For which initial conditions and on which intervals do we have unique solutions to the equation? In each case, what is the form of the solution?

Problem 4

Problem 4

Problem 4

Problem 5

(a) (10 points) Find the eigenvalues and eigenfunctions of the Laplacian on the square with periodic boundary conditions:

$$u_{xx} + u_{yy} + \lambda u = 0 \qquad 0 \le x \le 2\pi, 0 \le y \le 2\pi \tag{3}$$
$$u(0, y) = u(2\pi, y) \qquad 0 \le y \le 2\pi \tag{4}$$
$$u(x, 0) = u(x, 2\pi) \qquad 0 \le x \le 2\pi \tag{5}$$

(b) (10 points) Solve the heat equation on the square with periodic boundary conditions

$$u_t = u_{xx} + u_{yy} \qquad 0 \le x \le 2\pi, 0 \le y \le 2\pi \tag{6}$$
$$u(t, 0, y) = u(t, 2\pi, y) \qquad 0 \le y \le 2\pi \tag{7}$$
$$u(t, x, 0) = u(t, x, 2\pi) \qquad 0 \le x \le 2\pi \tag{8}$$

and initial data $u(0, x, y) = f(x, y)$.

Problem 5

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