AIM Qualifying Review Exam in Differential Equations & Linear Algebra

August 30, 2014

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

(a) Two matrices $A, B \in \mathbb{C}^{m\times m}$ are *unitarily equivalent* if $A = QBQ^*$ for some unitary $Q \in \mathbb{C}^{m\times m}$. Is it true or false that A and B are unitarily equivalent if and only if they have the same singular values? Show the steps in your reasoning.

(b) Compute the determinant of $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 4 & 0 & 2 & 0 \\ 0 & 5 & 0 & 3 \\ 0 & 0 & 6 & 0 \end{pmatrix}$.

Problem 1

Problem 1

Problem 1

Problem 2

(a) Let u be a vector of unit length ($\|u\|_2 = 1$) in $\mathbb{R}^n$ and I the $n \times n$ identity matrix. $H = I - 2uu^T$ is called a Householder reflection matrix because it reflects a given vector in the subspace of $\mathbb{R}^n$ orthogonal to u. Show that $\|Hw\|_2 = \|w\|_2$ for any $w \in \mathbb{R}^n$.

(b) Let $B = \begin{pmatrix} 9 & 2 \\ -1 & 12 \end{pmatrix}$ and $v = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$. Find $\lim\limits_{k \to \infty} \dfrac{B^k v}{\|B^k v\|_2}$.

Problem 2

Problem 2

Problem 2

Problem 3

Find the solution of the initial value problem $y'' - y = g(t), y(0) = 0, y'(0) = 1$.

Your solution should include a convolution integral with g and a known function. Note: The convolution integral may be written in the form $\int_a^b f(t-\tau)g(\tau)d\tau$ for particular values of a and b.

Problem 3

Problem 3

Problem 3

Problem 4

(a) Find the general solution of the ODE

$y'' + 4y' + 4y = e^{-2x}$.

(b) Solve Laplace's equation $\nabla^2 u(r, \theta) = 0$ inside the unit disk with the boundary condition

$u(1, \theta) = 3 \cos \theta + \cos 2\theta$.

Problem 4

Problem 4

Problem 4

Problem 5

Consider the wave equation modified by an additional term: $\partial_{tt}u = \partial_{xx}u - ku$.

(a) Assume $k > 0$. Solve the equation on $\{0 \leq x \leq \pi;\ t > 0\}$ subject to
the initial conditions $u(x, 0) = \sin(x),\ \partial_t u(x, 0) = 0$
and boundary conditions $u(0, t) = u(\pi, t) = 0$.

(b) Now assume $k < 0$. Explain how the solution changes as k decreases from 0 towards $-\infty$, noting any values of k where the behavior changes qualitatively.

Problem 5

Problem 5

Problem 5