AIM Preliminary Exam: Differential Equations & Linear Algebra

August 31, 2013

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1 Let A be an $n \times m$ real matrix, $n \geq m$. Let σ_i denote the singular values of A arranged in a non-increasing order, and $\vec{u}_i$, $\vec{v}_i$ denote the corresponding left and right singular vectors, respectively.

(a) State the singular value decomposition of A in terms of σ_i, $\vec{u}_i$, $\vec{v}_i$.

(b) Use part (a) to prove that for every vector $\vec{v} \in \mathbb{R}^m$, one has

$$\sigma_m \|\vec{v}\| \leq \|A\vec{v}\| \leq \sigma_1 \|\vec{v}\|.$$

Problem 1

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Problem 2 Suppose A and B are symmetric real matrices. Which of the following are necessarily symmetric?

If so, prove; if not, give a counterexample for A and B.

(a) $A^2 - B^2$

(b) $(A + B)(A - B)$

(c) ABA

(d) $ABAB$

Problem 2

Problem 2

Problem 2

Problem 3 Let f be a continuous functions on an interval I and consider the ordinary differential equation

$$y''(t) + f(t)y'(t) + y(t) = 0, \qquad t \in I.$$

Suppose that $y_1(t)$ and $y_2(t)$ are a fundamental set of solutions of this equation. Prove that between any consecutive zeroes t_1 and t_2 of $y_1(t)$ there exists a unique zero of $y_2(t)$.
Hint: If $y_2(t)$ has no zero between t_1 and t_2 use Rolle's theorem for the function $\frac{y_1(t)}{y_2(t)}$ and work toward a contradiction.

Problem 3

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Problem 4 Let Ω be an open, bounded and simply connected domain in $\mathbb{R}^3$, with smooth boundary $\partial\Omega$.

Suppose that $u : \overline{\Omega} \to \mathbb{R}$ is a twice continuously differentiable function satisfying the Laplace equation

$$\Delta u(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega$$

with boundary condition

$$\mathbf{n}(\mathbf{x}) \cdot \nabla u(\mathbf{x}) + h(\mathbf{x})u(\mathbf{x}) = 0, \quad \mathbf{x} \in \partial\Omega,$$

for a strictly positive function h defined on $\partial\Omega$. Here $\overline{\Omega} = \Omega \cup \partial\Omega$ and $\mathbf{n}$ is the outer normal at the boundary. Prove that u must be identically zero in $\overline{\Omega}$.

Problem 4

Problem 4

Problem 4

<u>Problem 5</u> Solve the first order partial differential equation

$$\partial_t u(x,t) + t^2 \partial_x u(x,t) = u(x,t),$$

for $x > 0$, with initial condition $u(x,0) = h(x)$ at $t = 0$, where h is a given continuously differentiable function defined on $\mathbb{R}$.

Problem 5

Problem 5

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