

# AIM Preliminary Exam: Differential Equations & Linear Algebra

*September 3, 2011*

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

**Problem 1**

- (a) Let  $\mathbf{a}, \mathbf{b} \in \mathbb{C}^n$ . Find the range, nullspace, rank, and nullity of the  $n \times n$  matrix  $\mathbf{M} = \mathbf{a}\mathbf{b}^\dagger$  (“ $\dagger$ ” denotes conjugate transpose).
- (b) Let  $\mathbf{a}, \mathbf{b} \in \mathbb{C}^n$ . Show that

$$\det(\mathbf{I} + \mathbf{a}\mathbf{b}^\dagger) = 1 + \mathbf{b}^\dagger\mathbf{a}$$

where  $\mathbf{I}$  is the  $n \times n$  identity matrix. (Hint: Among the many formulas for the determinant, one is more useful here than others).

- (c) Let  $\mathbf{A}$  be an  $n \times n$ , non-singular matrix. Show that some ordering of its rows leaves no zeros on the diagonal. (Hint: Among the many formulas for the determinant, one is more useful here than others).

Problem 1

Problem 1

Problem 1

## **Problem 2**

Let  $\mathbf{A}$  be an  $n \times n$  real, symmetric matrix. Let  $\mathbf{B}$  be the corresponding matrix with elements

$$B_{ij} := |A_{ij}|.$$

- (a) Prove that  $\mathbf{B}$  has an eigenvector with non-negative elements corresponding to its largest eigenvalue.
- (b) The spectral radius  $\rho(\mathbf{C})$  of a square matrix  $\mathbf{C}$  is defined to be the maximum of the magnitudes of its eigenvalues. Show that for a real symmetric matrix  $\mathbf{C}$ ,

$$\rho(\mathbf{C}) = \|\mathbf{C}\|$$

where  $\|\mathbf{C}\|$  denotes the 2-norm of the matrix, *i.e.*

$$\|\mathbf{C}\| := \max_{\|\mathbf{x}\|=1} \|\mathbf{C}\mathbf{x}\|$$

where  $\|\mathbf{x}\|$  denotes the usual Euclidean norm on vectors  $\mathbf{x}$ .

- (c) Show that the spectral radii of the matrices  $\mathbf{A}$  and  $\mathbf{B}$  from part (a) satisfy

$$\rho(\mathbf{B}) \leq \sqrt{n}\rho(\mathbf{A}).$$

(Hint: Given any vector  $\mathbf{v}$  with  $\|\mathbf{v}\| = 1$ , bound the magnitude of each component  $(\mathbf{B}\mathbf{v})_i$  of the vector  $\mathbf{B}\mathbf{v}$  in terms of  $\|\mathbf{A}\|$ . Sum, and use part (b)).

Problem 2

**Problem 2**



**Problem 2**

**Problem 3**

- (a) Let  $V$  be an  $n$ -dimensional inner product space over the real numbers  $\mathbb{R}$ . Let  $\mathbf{u}_1, \dots, \mathbf{u}_d$  be given vectors in  $V$ . Prove that the *Gram matrix*  $\mathbf{G}$  corresponding to these vectors,

$$G_{ij} = \langle \mathbf{u}_i, \mathbf{u}_j \rangle,$$

is always symmetric and positive semidefinite. Prove that it is also *positive definite* if and only if  $\mathbf{u}_1, \dots, \mathbf{u}_d$  are linearly independent.

- (b) Let  $\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_d \in \mathbb{R}^N$  be  $d+1$  points (vectors) in  $\mathbb{R}^N$ . Let  $d_{i,j} = \|\mathbf{p}_i - \mathbf{p}_j\|$  be the Euclidean distances between them. Show that the quadratic form

$$F(\mathbf{x}) = \sum_{i=1}^d \sum_{j=1}^d \left( d_{i,0}^2 + d_{j,0}^2 - d_{i,j}^2 \right) x_i x_j$$

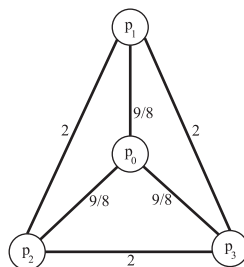
defined on vectors  $\mathbf{x} \in \mathbb{R}^d$  is positive definite. (Hint: Express  $d_{i,0}^2 + d_{j,0}^2 - d_{i,j}^2$  in terms of a Euclidean inner product and apply part (a), for example).

- (c) Let  $p_0, \dots, p_3$  be the points of an abstract metric space consisting of only four points. Denote the distance between the  $i$ -th and the  $j$ -th point by  $d_{i,j}$ . Let these distances be given by:

$$\begin{aligned} d_{0,i} &= d_{i,0} = \frac{9}{8} \text{ for all } i \in \{1, 2, 3\}, \\ d_{i,j} &= 2 \text{ for all } (i, j) \in \{1, 2, 3\} \times \{1, 2, 3\} \text{ and } i \neq j, \\ d_{j,j} &= 0 \text{ for any } j \in \{0, 1, 2, 3\} \text{ (obviously!)}. \end{aligned}$$

See the figure below illustrating the distances between the points. These distances indeed define a metric space, as they satisfy the triangle inequality (no need to show this).

Prove that this abstract metric space cannot be a subset of any Euclidean space  $\mathbb{R}^N$ . In other words, show that there do not exist points (vectors)  $\mathbf{p}_0, \dots, \mathbf{p}_3$  in any  $\mathbb{R}^N$  such that the Euclidean distances  $\|\mathbf{p}_i - \mathbf{p}_j\|$  are given by the  $d_{i,j}$  listed above.



Remark: One says that such an abstract metric space cannot be isometrically embedded in any Euclidean space. The condition of part (b) turns out to be sufficient (as well as necessary, as you will have shown) for embeddability.

**Problem 3**

**Problem 3**

**Problem 3**

**Problem 4**

Consider the ODE system

$$\begin{aligned}\frac{dx}{dt} &= 3x(1 - x - y) \\ \frac{dy}{dt} &= y\left(\frac{3}{2} - y - x\right).\end{aligned}$$

- (a) Find all critical points.
- (b) For each critical point, determine its type and stability, if possible, from the corresponding linearized system, and carefully sketch the trajectories in a neighborhood of each critical point.
- (c) Find the nullclines of the system, and determine the general direction of trajectories in the various regions that they define in the closed first quadrant ( $x \geq 0$  and  $y \geq 0$ ).
- (d) Using your results from parts (a), (b), and (c), carefully sketch the phase portrait of the system in the first quadrant. Describe the limiting behavior of solutions as  $t \rightarrow \infty$  for all initial conditions in the first quadrant.

Problem 4

Problem 4



Problem 4

**Problem 5**

Consider the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + \sin(t) \sin(2\pi x)$$

in the interval  $x \in (0, 1)$  and for  $t > 0$ , subject to the boundary conditions

$$u(0, t) = u(1, t) = 0, \quad \text{for all } t > 0,$$

and the initial conditions

$$u(x, 0) = \frac{\partial u}{\partial t}(x, 0) = 0, \quad \text{for } x \in (0, 1).$$

Here,  $c$  is a non-zero real constant (the wave speed).

- (a) Find the solution of this problem for all values of  $c > 0$ .
- (b) Describe how qualitative properties of the solution vary with  $c$ .

**Problem 5**

**Problem 5**

**Problem 5**