AIM Preliminary Exam: Differential Equations & Linear Algebra

September 4, 2010

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

Consider a particle of unit mass moving in one dimension with displacement $x(t)$. It is often said that one can think heuristically of Newton's equation with restoring force $-V'(x)$

$$\frac{d^2x}{dt^2} = -V'(x) \tag{1}$$

as describing the *horizontal* coordinate of a unit-mass bead sliding under gravity (with gravitational constant $g = 1$) along a frictionless wire in two dimensions (that is, lying in the vertical (x, h)-plane) bent into the shape of the graph $h = V(x)$ of the potential function $V(x)$.

But this assertion is false! In fact, the correct differential equation for $x(t)$ in the latter situation is

$$\frac{d^2x}{dt^2} = -\frac{V'(x)}{1+V'(x)^2}\left[1 + V''(x)\left(\frac{dx}{dt}\right)^2\right]. \tag{2}$$

(a) The position vector locating the bead at time t is $(x, h) = (x(t), V(x(t)))$, so the velocity vector is $(1, V'(x(t))) \cdot dx/dt$. The kinetic energy is therefore

$$K = \frac{1}{2}(1 + V'(x(t))^2)\left(\frac{dx}{dt}\right)^2$$

and the gravitational potential energy is $P = V(x(t))$. Show that (2) implies that the total energy $E := K + P$ is independent of time.

(b) Write the differential equation (2) as a first-order nonlinear system.

(c) Supposing for concreteness that $V(x) = x - x^3/3$, find all fixed points of the system from part (b).

(d) Supposing $V(x) = x - x^3/3$, determine the type of each of the fixed points from part (c) by linearization. What can you deduce about the behavior of the nonlinear system, and why?

(e) Supposing $V(x) = x - x^3/3$, carefully sketch the phase portrait of the system from part (b).

(f) Repeat parts (b)–(e) for the simpler system (1) and use your results to explain why the bead-on-wire analogy might be meaningful to describe Newton's equation after all.

Problem 1

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Problem 2

Let $\mathbf{A}$ be a real symmetric $n \times n$ matrix, and let $\mathbf{B}$ denote the square submatrix of $\mathbf{A}$ obtained by omitting the last row and last column. If $\mathbf{S}$ denotes the matrix with n rows and $n-1$ columns whose last row is all zeros and whose first $n-1$ rows make up the $(n-1)\times(n-1)$ identity matrix, then $\mathbf{B} = \mathbf{S}^{\mathsf{T}}\mathbf{A}\mathbf{S}$, and $\mathbf{S}^{\mathsf{T}}\mathbf{S}$ is the $(n-1)\times(n-1)$ identity matrix.

(a) Prove that the eigenvalues of any real symmetric matrix are real numbers, and any eigenvector can be scaled to have real elements.

(b) Let $a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_{n-1}$ be the eigenvalues of $\mathbf{A}$ with corresponding real orthonormal eigenvectors $\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{n-1}$, and let $b_1 \leq b_2 \leq \cdots \leq b_{n-1}$ be the eigenvalues of $\mathbf{B}$ with correpsponding real orthonormal eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1}$. Let $U_k \subset \mathbb{R}^n$ denote the real span of $\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_k$ and let $V_k \subset \mathbb{R}^{n-1}$ denote the real span of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$. Prove that

- If $\mathbf{w} \in V_k$ then $\mathbf{w}^{\mathsf{T}}\mathbf{B}\mathbf{w} \leq b_k \mathbf{w}^{\mathsf{T}}\mathbf{w}$, and
- If $\mathbf{w} \in U_{k-1}^{\perp}$ (that is, $\mathbf{w} \in \mathbb{R}^n$ is orthogonal to every vector in U_{k-1}) then $\mathbf{w}^{\mathsf{T}}\mathbf{A}\mathbf{w} \geq a_k \mathbf{w}^{\mathsf{T}}\mathbf{w}$.

(c) Prove that for all $k = 1, 2, 3, \ldots, n-1$, there exists a nonzero vector $\mathbf{w}$ such that both $w \in V_k$ and also $\mathbf{S}\mathbf{w} \in U_{k-1}^{\perp}$.

(d) Prove that $b_k \geq a_k$ for $k = 1, 2, 3, \ldots, n-1$.

Problem 2

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Problem 3

Let $\mathbf{A}$ be an $n \times n$ complex matrix whose eigenvalue with the largest magnitude lies strictly within the unit circle. Recall the 2-norm of a matrix:

$$\|\mathbf{A}\|_2 := \sup_{\mathbf{x} \neq \mathbf{0}} \frac{\|\mathbf{A}\mathbf{x}\|_2}{\|\mathbf{x}\|_2} = \sup_{\|\mathbf{x}\|_2 = 1} \|\mathbf{A}\mathbf{x}\|_2,$$

where for a vector $\mathbf{x} \in \mathbb{C}^n$, $\|\mathbf{x}\|_2 := \sqrt{|x_1|^2 + \cdots + |x_n|^2}$. For each of the following statements, determine whether true or false, and if true give a proof while if false give a counterexample:

(a) $\|\mathbf{A}\|_2 < 1$.

(b) $\lim_{n\to\infty} \|\mathbf{A}^n\|_2 = 0$.

Hint: consider the Jordan canonical form of $\mathbf{A}$.

Problem 3

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Problem 4

(a) Let $\mathbf{A}$ be the matrix

$$\mathbf{A} := \begin{bmatrix} -1 & -2 & 1 \\ 0 & -4 & 3 \\ 0 & -6 & 5 \end{bmatrix}.$$

Find the matrix $e^{t\mathbf{A}}$ and use it to find the solution of the initial-value problem

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t), \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(b) Let $u_1(x)$ and $u_2(x)$ be solutions of the non-constant coefficient differential equation $u''(x) + xu'(x) + \cos(x)u(x) = 0$ satisfying $u_1(0) = u_2'(0) = 1$ and $u_1'(0) = u_2(0) = 0$. Are $u_1(x)$ and $u_2(x)$ linearly independent functions on $\mathbb{R}$? Explain.

Problem 4

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<u>Problem 4</u>

Problem 5

Let $\{\alpha_n\}_{n=1}^{\infty}$ denote the positive solutions of the equation $\cot(\alpha) = \alpha$. Find the solution $u(x, y)$ of the boundary-value problem

$$\begin{cases} \Delta u = 0, & 0 < x < 1, \quad y > 0, \\ u_x(0, y) = 0, & y > 0, \\ u(x, 0) = 100, & 0 < x < 1, \\ u(1, y) + u_x(1, y) = 0, & y > 0 \end{cases}$$

that is bounded as $y \to \infty$.

Problem 5

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