

AIM Qualifying Review Exam in Differential Equations & Linear Algebra

January 2, 2017

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

- (a) (5 points) Construct a 2-by-2 matrix \mathbf{A} that maps $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
- (b) (15 points) Construct a 2-by-2 matrix \mathbf{B} that maps $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ to itself and has only one independent eigenvector.

Problem 1

Problem 1

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Problem 2

(20 points) Consider the linear system of equations for (x, y, z) :

$$px + y + z = 1$$

$$x + py + z = q$$

$$x + y + pz = r.$$

What conditions do the parameters (p, q, r) need to satisfy in order for the system to have

- (a) one and only one solution?
- (b) no solution?
- (c) more than one solution?

Problem 2

Problem 2

Problem 2

Problem 3

- (a) (10 points) Find the general solution $u(x)$ for the equation $u'' - 4u = 1 + e^{2x}$.
- (b) (10 points) Show that all eigenvalues of $y'' + \lambda y = 0, y'(0) = 0, y(1) = 0$ are non-negative.

Problem 3

Problem 3

Problem 3

Problem 4

(a) (10 points) Construct a Liapunov function for the system

$$\begin{aligned}x' &= -x^3 + xy^2 \\y' &= -2x^2y - y^3,\end{aligned}$$

and use it to show that the origin is a strictly stable critical point. Hint: consider polynomials.

(b) (10 points) Consider a 2D autonomous system

$$\begin{aligned}dx/dt &= f_1(x, y), \\dy/dt &= f_2(x, y),\end{aligned}$$

that satisfies $\partial_x f_1 + \partial_y f_2 > 0$ for all x and y . Show that such a system has no t -periodic solutions.

Problem 4

Problem 4

Problem 4

Problem 5

- (a) (10 points) Let $u(x, y)$ solve a Poisson problem in a closed 2D region Ω with Neumann conditions on the boundary:

$$\begin{aligned} -\Delta u &= f(x, y), \{x, y\} \in \Omega \\ \partial_n u &= g(x, y), \{x, y\} \in \partial\Omega. \end{aligned}$$

Show that the solution is unique up to an additive constant.

- (b) (10 points) Consider the following wave equation with initial and boundary conditions

$$\begin{aligned} \partial_{tt}w - \partial_{xx}w &= 0, 0 < x < 1, t > 0 \\ w(x, 0) &= \sin 2\pi x \\ \partial_t w(x, 0) &= x - x^2 \\ w(0, t) = w(1, t) &= 0. \end{aligned}$$

Find the value of $\int_0^1 \partial_t w^2 + \partial_x w^2 dx$ at $t = 1$.

Problem 5

Problem 5

Problem 5