AIM Preliminary Exam: Advanced Calculus & Complex Variables

January 5, 2015

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

Let

$$f(z) = \frac{(1-2z)\cos(2\pi z)}{z^2(1-z)^2}$$

for $z \in \mathbb{C} - \{0, 1\}$. Show that there is an analytic (single valued) function $F(z)$ on $\mathbb{C} - \{0, 1\}$ such that $F' = f$.

Problem 1

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Problem 2

Calculate the following integral using complex variable methods:

$$\int_0^\infty \frac{\log(1+x^2)}{x^{1+p}}\,dx$$

where $0 < p < 2$.

Hint: Try integrating by parts.

Problem 2

Problem 2

Problem 2

Problem 3

Let $x_1, x_2, \ldots, x_n$ be positive numbers. Prove that

$$\frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots \frac{x_{n-1}}{x_n} + \frac{x_n}{x_1} \geq n.$$

Problem 3

Problem 3

Problem 3

Problem 4

(a) Show that

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^p}$$

converges for all x except multiples of 2π, provided that $p \in (0, 1]$.

(b) Determine convergence (or not) of the following sum for all values of the parameter $p \in \mathbb{R}$:

$$\sum_{n=2}^{\infty} \frac{1}{(\log n)^p}.$$

(c) Determine whether the following sum converges:

$$\sum_{n=2}^{\infty} \frac{1}{(\log n)^{\log n}}.$$

Problem 4

Problem 4

Problem 4

Problem 5

Let $u \in C^2(\mathbb{R}^d)$ be harmonic:

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_d^2} = 0$$

for all $x \in \mathbb{R}^d$.

(a) Show that u has the following *mean value property*:

$$\frac{1}{\text{Area}(B_r(x))} \int_{\partial B_r(x)} u(y)\, d\sigma(y) = u(x)$$

for any $x \in \mathbb{R}^d$ and $r > 0$. Here, $B_r(x)$ denotes the ball of radius r centered at x; $\partial B_r(x)$ denotes its boundary, the sphere; Area$(\partial B_r(x))$ denotes its surface area; and $d\sigma$ denotes the surface area element. In words, the formula says that average of a harmonic function u over the surface of a ball is the value of u at the center. You may take $d = 3$, if you like.

Hint: Start as follows:

$$\frac{d}{dr} \frac{1}{r^{d-1}} \int_{\partial B_r(x)} u(y)\, d\sigma(y) = \frac{d}{dr} \int_{\partial B_1(0)} u(x + ry)\, d\sigma(y) = \dots$$

(b) Using part (a), show that u also has the following mean value property:

$$\frac{1}{\text{Vol}(B_r(x))} \int_{B_r(x)} u(y)\, dy = u(x)$$

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Problem 5

Problem 5

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