AIM Preliminary Exam: Advanced Calculus & Complex Variables

January, 2019

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

(20 points) Consider a sequence $\{a_n\}_{n\geq 1}$ of numbers satisfying $a_n > 0$ and converging to 0. Prove that for all integers $r \geq 1$, there are infinitely many n such that

$$a_n > a_{n+r}. \tag{1}$$

Problem 1

Problem 1

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Problem 2

(20 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function. Prove that the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = f(x + 5) - f(x)$ is bounded.

Problem 2

Problem 2

Problem 2

Problem 3

(20 points) Let $a, b \in \mathbb{C}$, with $|b| < 1$. Calculate the integral

$$\frac{1}{2\pi i} \int_{|z|=1} \frac{|z-a|^2}{|z-b|^2} \frac{dz}{z}. \tag{2}$$

Problem 3

Problem 3

Problem 3

Problem 4

(20 points) Let $f(z)$ be an entire function which satisfies $|f(1/n)| = 1/n^2$ for all integer $n \geq 1$, and $|f(i)| = 2$.

(i) Prove that $\overline{f(\overline{z})}$ is entire.

(ii) What are the possible values of $|f(-i)|$?
Hint: You may try to make use of the function in question (i) and recall the following uniqueness result for analytic functions: Let $g(z)$ and $h(z)$ be two analytic functions in some domain Ω. Suppose that $g(z) = h(z)$ for all points of a subset of Ω that has limit point in Ω. Then, $g(z) = h(z)$ in Ω.

Problem 4

Problem 4

Problem 4

Problem 5

(20 points)

(i) Find the points of intersection of the sphere centered at the origin in $\mathbb{R}^3$, of radius $r = 2\sqrt{2}$, and the line passing through the points $\vec{\mathbf{a}} = (1, 2, 3)$ and $\vec{\mathbf{b}} = (0, 2, 2)$.

(ii) Determine the surface S of all points in $\mathbb{R}^3$ that are at the same distance from $(0, -2, 0)$ and $(2, 2, 2)$.

(iii) Determine the distance between the planes:

$$P_1 = \{(x, y, z) \in \mathbb{R}^3 \ \text{ so that } \ x + 2y - z = -1\},$$
$$P_2 = \{(x, y, z) \in \mathbb{R}^3 \ \text{ so that } \ 3x + 6y - 3z = 3\}.$$

Problem 5

Problem 5

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