AIM Qualifying Review Exam in Advanced Calculus & Complex Variables

September 5, 2015

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

Let $f(z)$ be an entire function and $h \neq 0$. Find *all* possible values of the contour integral

$$\frac{1}{2\pi i} \int_C \frac{f(z)}{z(z^2 - h^2)} \, dz.$$

The contour C is allowed to vary subject to the following assumptions:

(a) C does not pass through $0, \pm h$.

(b) The orientation of C is counterclockwise.

(c) C is a smooth simple closed curve.

Problem 1

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Problem 2 Prove that the polynomial equation

$$z^5 + \frac{z^3}{4} + \frac{z^2}{8} + \frac{z}{16} + \frac{1}{32} = 0$$

has 5 roots inside the circle $|z| = 1$.

Problem 2

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Problem 3

Suppose that $0 < x < y$.

(a) If $0 < \alpha < 1$, prove that

$$\frac{\log\left((1-\alpha)x+\alpha y)\right) - \log x}{((1-\alpha)x+\alpha y) - x} > \frac{\log y - \log\left((1-\alpha)x+\alpha y)\right)}{y - ((1-\alpha)x+\alpha y)}.$$

(b) Recast the above inequality as

$$\log\left((1-\alpha)x+\alpha y\right) \geq (1-\alpha)\log x + \alpha \log y.$$

Interpret this inequality using a graph of $\log x$.

(c) If $k > 0$, prove that

$$\int_k^{k+1} \log x \, dx \geq \frac{\log k + \log(k+1)}{2}$$

(d) If n is a positive integer, prove that

$$\log n! \geq \int_1^n \log x \, dx \geq \log n! - \frac{\log n}{2}.$$

Problem 3

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Problem 3

Problem 4

Suppose that $f : [0, a] \to \mathbb{R}$ is a continuous function.

(a) Argue that the iterated integral

$$\int_0^a \int_x^a \int_y^a f(x)f(y)f(z)\,dz\,dy\,dx$$

is equal to the triple integral

$$\int f(x)f(y)f(z)\,dx\,dy\,dz$$

over the region $0 \le x \le y \le z \le a$.

(b) Argue that the value of the triple integral does not change for the region $0 \le y \le x \le z \le a$, or any of the 4 additional regions obtained by taking other permutations of x, y, z.

(c) Prove that the value of the iterated integral is equal to

$$\frac{1}{6}\left(\int_0^a f(x)\,dx\right)^3.$$

Problem 4

Problem 4

Problem 4

Problem 5

Consider the power series

$$z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \cdots$$

(a) Prove that the power series converges for $z = -1$.

(b) Prove that the power series converges for $z = i$.

(c) Prove that the power series converges for all complex z with $|z| = 1$ except $z = 1$.

Problem 5

Problem 5

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