AIM Preliminary Exam: Advanced Calculus & Complex Variables

September 1, 2012

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

(a) Let $w \mapsto w^a$ denote the principal branch of the power function defined for $|\arg(w)| < \pi$. Find and sketch the set of all values of $z \in \mathbb{C}$ for which the following identity holds for all $a \in \mathbb{R}$:

$$(z^2 - 1)^a = (z+1)^a (z-1)^a.$$

(b) Evaluate the contour integral

$$\oint_C \frac{\log(z)\, dz}{(2z-1)^2}$$

where $\log(z)$ is the principal branch and C is the positively-oriented unit circle.

Problem 1

Problem 1

Problem 1

Problem 2

Consider the integral with complex parameter z given by

$$f(z) := \int_0^{+\infty} \frac{e^{-t} - e^{-zt}}{t}\, dt.$$

(a) Prove that the integral converges absolutely for $\Re\{z\} > 0$. (As the integrand is analytic in z for each $t > 0$, this implies that $f(z)$ is an analytic function of z for $\Re\{z\} > 0$.)

(b) Find a simple expression for the analytic function with which $f(z)$ agrees for $|z - 1| < 1$.

Problem 2

Problem 2

Problem 2

Problem 3

(a) Find the radius of convergence (about $x = 0$) of the series

$$g(x) := \sum_{n=0}^{\infty} (n^2 + n)x^n.$$

(b) Evaluate the numerical series

$$\sum_{n=0}^{\infty} \frac{n^2 + n}{2^n}.$$

(c) Evaluate the limit

$$\lim_{n\to\infty} \sum_{j=1}^{n} \frac{1}{n + jc}, \quad c > -1.$$

Problem 3

Problem 3

Problem 3

Problem 4

A function $f : [a, b] \to \mathbb{R}$ is called *Hölder continuous with exponent* $\alpha > 0$ if there is a positive constant $M_f > 0$ such that whenever $a \leq x \leq y \leq b$,

$$|f(x) - f(y)| \leq M_f |x - y|^\alpha.$$

The set of all such functions is denoted $C^{0,\alpha}([a, b])$.

(a) Prove that if $f \in C^{0,\alpha}([a, b])$, then f is uniformly continuous on $[a, b]$.

(b) Prove that if $f \in C^{0,\alpha}([a, b])$ and $\alpha > 1$, then f is a constant function.

(c) Suppose that $0 < \alpha < 1$. Prove that the function $f(x) := x^\alpha$ belongs to $C^{0,\alpha}([0, 1])$.

Problem 4

Problem 4

Problem 4

Problem 5

Let C be the oriented curve lying on the unit sphere and given by the parametric equations

$$x = \cos(\theta)\sin(\tfrac{\pi}{3} + \tfrac{\pi}{6}\cos(7\theta)), \quad y = \sin(\theta)\sin(\tfrac{\pi}{3} + \tfrac{\pi}{6}\cos(7\theta)), \quad z = \cos(\tfrac{\pi}{3} + \tfrac{\pi}{6}\cos(7\theta))$$

with θ varying from $\theta = 0$ to $\theta = 2\pi$, as illustrated in the figure.

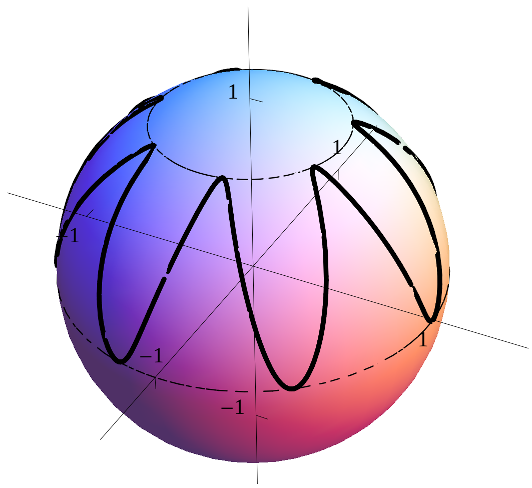

Figure 1: The oriented curve C.

Calculate the total circulation along C of the vector field

$$\mathbf{F}(x, y, z) := \begin{pmatrix} -2xz \\ 0 \\ y^2 \end{pmatrix}.$$

Problem 5

Problem 5

Problem 5