The Amazing Power of Dimensional Analysis in Finance: Market Impact and the Intraday Trading Invariance Hypothesis

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joint work with M. Pohl, A. Ristig, L. Tangpi

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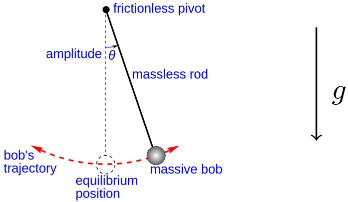
Dimensional Analysis

The period of the pendulum

Functional relation:

$$\text{period} = f(l, m, g).$$



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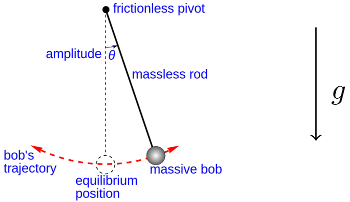

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- the length l of the pendulum, measured in meters: dimension $\mathbb{L}$.

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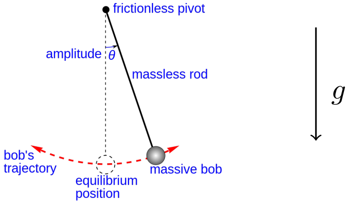

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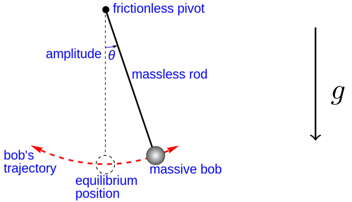

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- the length l of the pendulum, measured in meters: dimension $\mathbb{L}$.
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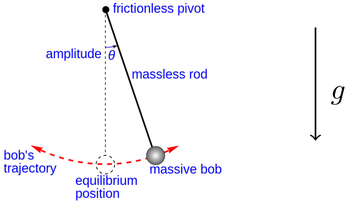

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Basic assumption: The three variables l, m, g **fully explain** the period of the pendulum.

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Ansatz:

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Unique solution: $y_1 = \frac{1}{2}$, $y_2 = 0$, $y_3 = \frac{1}{2}$,

$$\text{period} = \text{const} \cdot \sqrt{\frac{l}{g}}.$$

Remark: The Ansatz

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As the row vectors of the above matrix span $\mathbb{R}^3$, this fully determines the function F (up to a constant).

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Definition

The market impact G is the size of price change caused by a bet (in percentage of the price).

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Ansatz:

$$G = g(Q, P, V, \sigma^2) = \text{const} \cdot Q^{y_1} P^{y_2} V^{y_3} \sigma^{2y_4},$$

$Q=$ size of bet, $P=$price of share, $V=$traded daily volume, $\sigma=$volatility.

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Leads to **three** linear equations in **four** unknowns y_1, y_2, y_3, y_4.
The solution has one degree of freedom

$$G = \text{const} \cdot \left(\frac{Q\sigma^2}{V}\right)^y,$$

where $y \in \mathbb{R}$ and const > 0 are still free.

This time the ansatz **does** restrict the generality! The general solution for G, respecting the dimensional restrictions is

$$G = g\Big(\frac{Q\sigma^2}{V}\Big),$$

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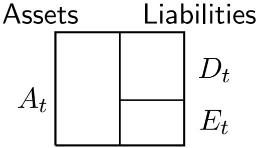
Can we find one more equation which will allow us to get a unique solution?

Kyle, Obizhaeva (2016): YES

Leverage neutrality

Theorem of Modigliani-Miller (1958):

A_t assets
D_t debt
E_t equity



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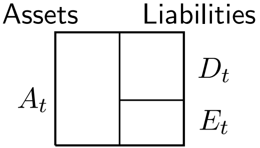

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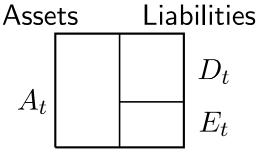

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Keeping the debt D_t constant, we therefore get $dA_t = dE_t$ so that

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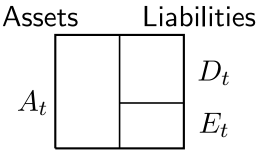

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Conclusion: Denoting by $L_t = \frac{A_t}{E_t}$ the *leverage* of the company, the relative dynamics of $(E_t)_{t\geq 0}$ are simply proportional to the leverage L_t.

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- P is multiplied by $\frac{1}{2}$.
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- G is multiplied by 2.
- Q, V remain unchanged.

Leverage neutrality: (Kyle, Obizhaeva, 2017)

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Mathematically speaking, the variation of the leverage (dimension $\mathbb{M}$) is analogous to the scalings of the dimensions $\mathbb{S}, \mathbb{U}$, and $\mathbb{T}$.

	Q	P	V	σ^2	G
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$\mathbb{U}$	0	1	0	0	0
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Theorem (Pohl, Ristig, S., Tangpi, '17 based on Kyle, Obizhaeva, '16):

Assume $G = g(Q, P, V, \sigma^2)$ is such that

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In particular, the market impact G is proportional to the *square root of the size* Q of the meta-order.

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Unfortunately it is hard (if not impossible) to analyze empirically the “true” market immpact G of an order size Q.

We can hardly observe the **meta-orders**, however we can observe the **actual orders**.

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Following Kyle and Obizhaeva (2017) and Bouchaud et al. (2016) the following quantities come into one's mind.

- V traded volume (per day), $[V] = \mathbb{S}\mathbb{T}^{-1}$
- P price of a share, $[P] = \mathbb{U}\mathbb{S}^{-1}$
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- C cost per trade, $[C] = \mathbb{U}$.

First attempt of explanatory variables: P, V, σ^2

Proposition:

Assume that the number of trades N depends *only* on the 3 quantities σ^2, P and V, i.e.,

$$N = g(\sigma^2, P, V),$$

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Too simplistic!

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Then, there is a constant $c > 0$ such that the number of trades N obeys the relation

$$N^{3/2} = c \cdot \frac{\sigma P V}{C}.$$

	P	V	σ^2	C	N
$\mathbb{S}$	-1	1	0	0	0
$\mathbb{U}$	1	0	0	1	0
$\mathbb{T}$	0	-1	-1	0	-1
$\mathbb{M}$	-1	0	2	0	0

Table: A labelled overview of the dimensions of P, V, σ^2, C and N.

Empirical Results

Our empirical analysis is based on limit order book data provided by the LOBSTER database (`https://lobsterdata.com`). The considered sampling period begins on January 2, 2015 and ends on August 31, 2015, leaving 167 trading days. Among all NASDAQ stocks, $d = 128$ sufficiently liquid stocks with high market capitalizations are chosen.

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N_j denotes the number of trades in the interval j,

Q_j $= N_j^{-1} \sum_{k=1}^{N_j} Q_{t_k}$ denotes the average size of the trades in the interval j, where Q_{t_k} denotes the number of shares traded at time t_k,

V_j $= N_j \times Q_j$ is the traded volume in the interval j,

P_j $= N_j^{-1} \sum_{k=1}^{N_j} P_{t_k}$ denotes the average midquote price in the interval j, where $P_{t_k} = (A_{t_k} + B_{t_k})/2$ and A_{t_k} (resp. B_{t_k}) denotes the best ask (resp. bid) price after the transaction at time t_k,

Empirical Results contd.

$\hat{\sigma}_j^2$ denotes the estimated squared volatility in the interval j,

S_j $= N_j^{-1} \sum_{k=1}^{N_j} S_{t_k}$ denotes the average bid-ask spread in the interval j, where $S_{t_k} = A_{t_k} - B_{t_k}$ is the bid-ask spread after the transaction at time t_k, and

C_j $= Q_j \times S_j$ is the spread cost per trade in the interval j.

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$$\log(N_{ij}) \sim \beta_i \log(\hat{\sigma}_{ij}^2) + \gamma_i \log\left(\frac{P_{ij}V_{ij}}{C_{ij}}\right) + \epsilon_{ij}.$$

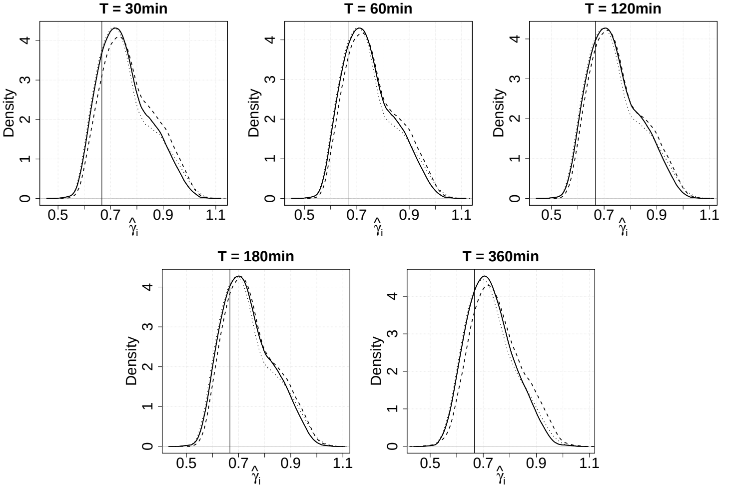


Figure: The panels show kernel density estimates across the estimated parameters $\hat{\gamma}_i$ for different interval lengths $T \in \{30, 60, 120, 180, 360\}$ min.

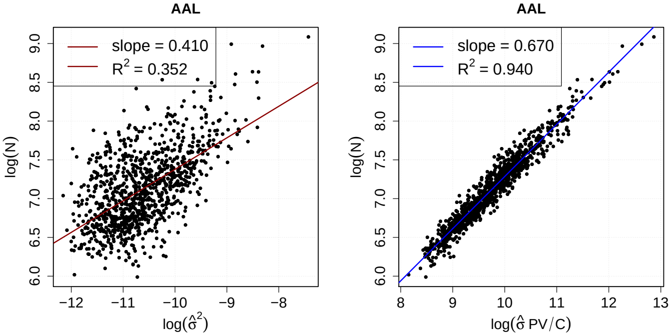
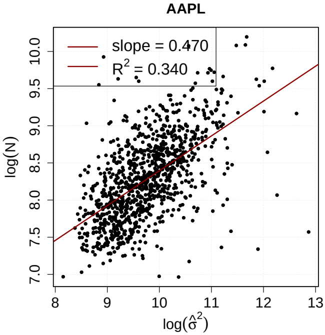
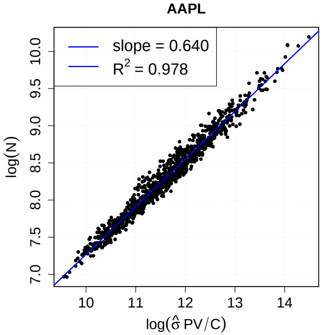


Figure: The dependent variable $\log N$ is plotted versus the explanatory variables $\log \hat{\sigma}$ resp. $\log(\hat{\sigma} PV/C)$ for fixed interval $T = 60$ min and the AAL stock. The lines indicate the estimated linear relations between the considered quantities.





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What does the empirical data tell us on this issue?

Typical estimator for σ^2:

$$\hat{\sigma}^2 := \sum_{k=1}^{n} \Big(\log(P_{t_k}) - \log(P_{t_{k-1}}) \Big)^2.$$

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But we could also consider, for $H \in]0,1[$,

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- market micro structure effects (Bouchaud, Rosenbaum, . . .)

$$\mathbb{E}[(W_{t+T} - W_t)^2]^{1/2} = T^{1/2},$$

but $\mathbb{E}[(\lceil W_{t+T} \rceil - \lceil W_t \rceil)^2]^{1/2} \sim T^{1/4}$, for $T \mapsto 0$.

Theorem [$(1+H)$-law] (Pohl, Ristig, S., Tangpi, 2018) : Suppose that
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$$N = g(\hat{\sigma}^2(H), P, V, C),$$

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	P	V	$\hat{\sigma}^2(H)$	C	N
$\mathbb{S}$	-1	1	0	0	0
$\mathbb{U}$	1	0	0	1	0
$\mathbb{T}$	0	-1	-2H	0	-1
$\mathbb{M}$	-1	0	2	0	0

Table: An overview of the dimensions of quantities $P, V, \hat{\sigma}^2(H), C$ and N.

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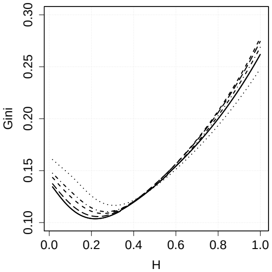

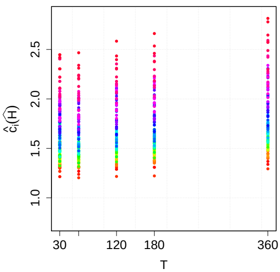

The left panel illustrates the Gini-coefficient in dependence of H for $T = 30$min (solid), $T = 60$min (long-dashed), $T = 120$min (dashed), $T = 180$min (dashed-dotted) and $T = 360$min (dotted).

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