Eigenvalues and random matrices

Jonathan Husson

Random matrices were incepted in the 1920s by the statistician John Wishart to study empirical covariances and then resurfaced in the 1950 when physicist Eugene Wigner used them to model heavy nuclei (for which he was awarded the Nobel Prize in 1963). Since then, the field has known an important expansion to a wide array of scientific fields from number theory to mathematical physics. Now, random matrices can be found in finance, in machine-learning and in the study of complex networks.

But what is a random matrix? Well, it is a matrix whose entries are chosen (*drum roll*) randomly. For instance a classical model of random matrices, *Wigner matrices* are a family of random $N \times N$ symetric matrices W_N that looks like this :

$$W_N = \frac{1}{\sqrt{N}} \begin{pmatrix} \sqrt{2}a_{1,1} & a_{1,2} & \cdots & a_{1,N} \\ a_{1,2} & \sqrt{2}a_{2,2} & \cdots & a_{2,N} \\ \vdots & \vdots & \ddots & \\ a_{1,N} & a_{2,N} & & \sqrt{2}a_{N,N} \end{pmatrix}$$

where all the entries $(a_{i,j})_{1 \leqslant i \leqslant j \leqslant N}$ are independent copies of the same random variable whose average is 0. In random matrix theory, we are often interested in the matricial properties of W_N (such as its eigenvalues, its eigenvectors or its determinant) when N becomes very large. For example, a question of particular interest is: how does the spectrum of such matrices globally behaves when N tends to infinity? It turns out for such a model, the asymptotic distribution of the eigenvalues is known (this result is called *Wigner's theorem*).

The goal of this project is to introduce its participants to the basics of the study of eigenvalues of random matrices. We will go through some classical random matrix models to illustrate some of the most celebrated results of the field. The approach will mix proofs and simulations depending on the students backgrounds and wishes. Then if time permits there are several directions the project may take, like the study of extremal eigenvalues of a perturbed random matrix or a foray into the world of free probability and its links to polynomials in random matrices.

Prerequisites:

- Some familiarity with probability theory (ideally 425: Introduction to probability or equivalent).
- 215 Multivariable and vector calculus or equivalent
- 217 Linear Algebra or equivalent
- Some coding experience (Matlab, Mathematica, C++, Python, etc...)