An introduction to optimal transport and its application to mathematical finance

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1 Introduction

This project first begins with a gentle introduction to the theory of optimal transport, which is the mathematical formulation of moving mass from a source to a destination in the most efficient manner. A particularly wonderful offspring of optimal transport theory is the so-called Wasserstein distance - a distance that measures how *far* two points are in terms of how *difficult* it is to optimally move from one point to another. This and many other advancements in this field has significant ramifications across many areas, such as partial differential equations (PDEs), geometry and machine learning. However, its application to mathematical finance may be limited due to a very natural reason - one cannot use tomorrow's information to plan today's transport. The exploration of these potential limitations and its resulting solution will form the second part of the project.

2 Objectives

The first part of the project will follow, depending on the students preference, either the more theoretical approach in Ambrosio et al. (2021) or more computation-based approach in Peyré and Cuturi (2020) to obtain basics of optimal transport theory. Then, the second part is concerned with some of the following areas, depending on students' progress and preference:

1. **Causal optimal transport (COT).** We consider a simple but important example highlighting the failure of using Wasserstein distances in the optimal stopping problem, a prototypical problem arising from mathematical finance. We will then turn our attention to its remedy, namely *causal* optimal transport, and discuss some theory.

2. **Applications to mathematical finance.** Students will explore some dynamic optimization problems arising from mathematical finance, such as the optimal stopping problem, mean-variance hedging, and log-utility maximization, as well as the application of COT to these problems.

3. With the basic theory established, students can embark on either of the following numerical investigations:

   - **Time Series Generation:** Time series generators are tools that help us generate synthetic time-dependent data in the case where data is scarce. There are many such generators available that promise good performance, including those utilizing COT. Can we compare these generators?
   - **Time Series Estimation:** Another possible line of investigation concerns the numerical estimation of the distribution of time-dependent data via the empirical, *adapted* empirical and *smooth adapted* empirical measures. Can we compare these estimators?

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3 Prerequisites

Probability theory (Math 525). Mathematics of finance (Math 423) and Introduction to stochastic analysis for finance (Math 474) are recommended but not required.

References

L. Ambrosio, E. Brué, D. Semola, et al. *Lectures on optimal transport*, volume 130. Springer, 2021.

G. Peyré and M. Cuturi. Computational optimal transport, 2020. URL https://arxiv.org/abs/1803.00567.