Lost, but Not Forever: Recurrence on Infinite Graphs

Alex Bzdula, Kenny Deng, Oliver Garnica, Chengzhi Zheng
Mentor: Yanlong Hao
University of Michigan Laboratory of Geometry

LOG(M)

ABSTRACT

The project is about the study of recurrence and transience of Brownian motion on Riemannian manifolds. Due to the complexities of the differential geometry, we will study simple random walks on graphs instead, since there is a direct correspondence between Brownian motion on Riemannian manifolds to simple random walks on graph, as discovered by [[1]]. Although the findings in the paper include more than even degree regular graphs, we limit our scope to those graphs since it is easier to reconstruct the corresponding manifolds explicitly.

Research Question: What properties of an infinite graph determine whether a random walk is transient or recurrent?

Understanding random walks on graphs, and through them Brownian motion on manifolds, builds a bridge between probability theory and differential geometry. This connection provides a probabilistic viewpoint on geometric and analytic properties such as curvature, heat diffusion, and topological structure.

THEORETICAL BACKGROUND

Graphs

Definition. *A graph is **k-regular** if each vertex is connected to k edges.*

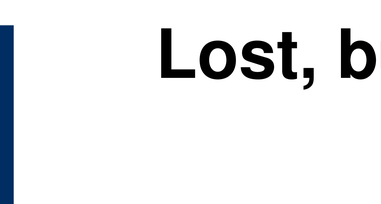


Figure 1: K_5 *is an example of a 4-regular graph.* $K_{3,3}$ *is an example of a 3-regular graph.*

Recurrence and Transience

Definition. *A **random walk** on a graph is a sequence of discrete steps on a graph where at each step the walker moves from its current vertex to a neighboring vertex according to some specified probability rule.*

Definition. *A random walk is **recurrent** if* $\sum_{n=1}^{\infty} p_n(v) = \infty$ *where* $p_n(v)$ *denotes the probability that the walk is at vertex* v *at time* n*. Analogously, a random walk is **transient** if* $\sum_{n=1}^{\infty} p_n(v) < \infty$*. This property is independent to the choice of* v*.*

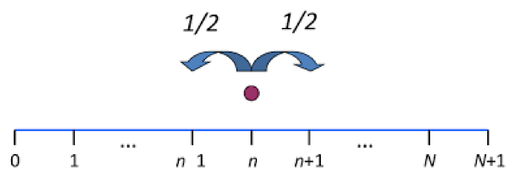
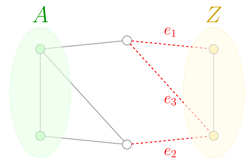


Figure 2: *Image of a random walk on the 1-Dimentional lattice* $\mathbb{Z}$ *(2-regular graph). The random walk on this graph is recurrent.*

Definition. *Let* A, Z *be subgraphs of* Γ*, a **cutset** separating* A *and* Z *is a set of edges s.t. every path from* A *to* Z *must contain an edge from the cutset.*



Theorem 1 (Nash-Williams Criterion)**.** *[[2]] Let* $\Gamma = (V, E)$ *be a* k*-regular graph. Let* u *be a vertex in* Γ*, and* $\{\Pi_i\}_{i=1}^{\infty}$ *be a sequence of disjoint, finite cutsets of* Γ *that seperates* u *from infinity. If*

$$\sum_n \frac{k}{|\Pi_n|} = \infty,$$

then the graph is recurrent.

RESULTS

Definition. *Let* $\Gamma := (V, E)$*, given a fixed* $v \in V$*, let* $V_n = \{u \in V \mid d(v, u) \leq n\}$ *and* $E_n = \{(u, u') | u, u' \in V_n\}$*. Define the **edge vertex ratio** at* v *as* $\lim_{n\to\infty} \frac{|E_n|}{|V_n|}$*.*

Proposition 2. *Let* Γ *be an infinite locally finite graph. If*

$$\sum_{n=1}^{\infty} \frac{n}{V_n} = \infty,$$

then Γ *is recurrent.*

Lemma 3. *For any integer* $k \geq 2$*, there exists an infinite recurrent graph with edge-vertex ratio* $k/2$*.*



Figure 3: *The graphs in this figure demonstrate the constructions satisfying Lemma 3. The triangular prism, rectangular prism, and heptagonal prism are 4-regular, 5-regular, 9-regular graph's respectively. Their edge-vertex ratio is* $\frac{4}{2}$*,* $\frac{5}{2}$*, and* $\frac{9}{2}$*. These graph's are all roughly isometric to* $\mathbb{Z}$*, hence recurrent.*



Definition. *Let* $\Gamma = (V, E)$*,* $v_0 \in V$*. Consider* $R_\Gamma = \{$*All infinite rays of same origin in* $\Gamma\}$ *and the equivalence relation on* R_Γ *of* $r_1 \sim r_2$ *provided that* $\forall$ *finite* $S \subseteq V$*,* r_1 *and* r_2 *will lie in the same component of* $V \setminus S$*. We define the **End** of the graph as* $R_\Gamma / \sim$

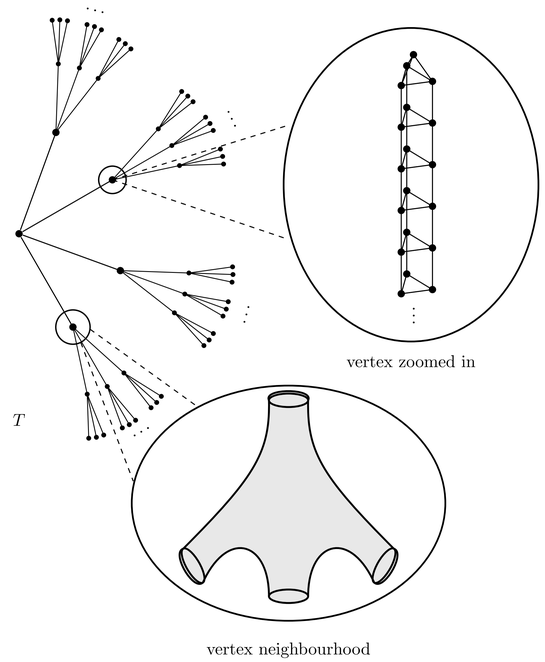


Figure 4: *4-regular ternary-tree model* T *with two constructions: a graph obtained by replacing each vertex with a tube, and the corresponding manifold construction.*

Theorem 4. *The graph depicted in Figure 4 is recurrent and has uncountably infinite ends.*

Remark 5. *The above graph is the* 4*-regular special case. The construction works for any* k*-regular graph,* $k \geq 3$*—by replacing repeating* K_3 *cross-sections in the branch to* K_{k-1}*. The edge vertex ratio is always* $k/2$*.*

FUTURE WORK

- Can the edge-vertex ratio be useful in combination with other invariants to determine recurrence/transience, even though it fails alone?
- Ultimately, can discrete graph-theoretic conditions fully capture the recurrence behavior of Brownian motion on the corresponding manifold?

REFERENCES

[1] Masahiko Kanai. *Rough isometries, and combinatorial approximations of geometries of non-compact riemannian manifolds.* Journal of the Mathematical Society of Japan, 1985.
XYZ Press, (2017)
Research Writinigs around Mathematics, vol. 239, pg 128-133, 1859.

[2] Russell Lyons and Yuval Peres. Probability on trees and networks, volume 42. *Cambridge University Press, 2017.*

[3] Yang Zhang, Edwin K. P. Chong, Jan Hannig, and Donald Estep. *On Continuum Limits of Markov Chains and Network Modeling.* Proceedings of the 49th IEEE Conference on Decision and Control (CDC), pp. 6779–6784, 2010.

Thank you, Yanlong Hao, for supporting our research project.