Cone with the Flow: Geometry in SO(2,1)

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ABSTRACT

There is a recent result of Kapovich–Leeb [1] on domains of discontinuity of Lorentzian affine transformations. In our project, we restrict to the 3-dimensional case :

$$\mathrm{Aff}(\mathbb{R}^3) = SO(2,1) \ltimes \mathbb{R}^3$$

acting on $\mathbb{R}^3$, and study the geometry of candidate fundamental domains. Our goal is to understand how the Lorentzian structure, the light cone, and the Margulis invariant interact with properness and with the geometry of crooked planes. In particular, we investigate what remains of the fundamental-domain when the Margulis invariant changes sign.

OBJECTIVE :

Analyze affine transformations $(A, u) \in \mathrm{SO}(2,1) \ltimes \mathbb{R}^3$ acting on $\mathbb{R}^3$, with emphasis on :

- the geometry preserved by $\mathrm{SO}(2,1)$,
- transformation group $\Gamma = \langle (A,u), (B,v) \rangle$ where A, B are diagonalizable $\mathrm{SO}(2,1)$ matrices with distinct eigenvalues,
- candidate fundamental domains bounded by crooked planes,
- the role of the Margulis invariant in determining proper and improper actions.

THEORETICAL BACKGROUND

Definition. Special Orthogonal Group SO(2,1)
On $\mathbb{R}^3$ (with coordinates (x,y,t)), define the bilinear form

$$\langle \vec{u}, \vec{v} \rangle_{2,1} = \vec{u}^T \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix} \vec{v} = x_1 x_2 + y_1 y_2 - t_1 t_2$$

The group $\mathrm{SO}(2,1)$ consists of all matrices A orthogonal with respect to $\langle \cdot, \cdot \rangle_{2,1}$:

$$A^T \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix} A = \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix} \;\Rightarrow\; \langle A\vec{x}, A\vec{x} \rangle_{2,1} = \langle \vec{x}, \vec{x} \rangle_{2,1}.$$

Any matrix A in $\mathrm{SO}(2,1)$ can be parameterized to

$$A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cosh\psi & 0 & \sinh\psi \\ 0 & 1 & 0 \\ \sinh\psi & 0 & \cosh\psi \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Definition. The Light Cone (Fig. 1)
The set $\mathcal{C} = \{u \in \mathbb{R}^3 \mid \langle u, u \rangle_{2,1} = 0\}$ is the **light cone**

The light cone is the set of vectors which have **norm 0**.
All points on the light cone are invariant under any transformation in $\mathrm{SO}(2,1)$
If a vector has negative norm, it is a **time-like** vector. If a vector has positive norm it is a **space-like** vector

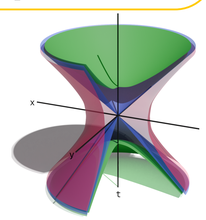


Figure 1:

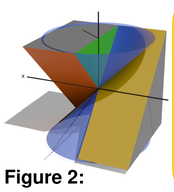


Figure 2:
Crooked half-space.

Definition. The Crooked Half-Space (Fig. 2)
A space partitioned by **4** half- or quarter-planes, relative to an isotropic basis $\{e_x, e_y, e_z\}$, such that one half contains :

$$\begin{Bmatrix} e_x > 0 \\ e_y > 0 \\ e_z > 0 \end{Bmatrix} \cup \begin{Bmatrix} e_x < 0 \\ e_y > 0 \\ e_z > 0 \end{Bmatrix} \cup \begin{Bmatrix} e_x < 0 \\ e_y < 0 \\ e_z > 0 \end{Bmatrix} \cup \begin{Bmatrix} e_x < 0 \\ e_y < 0 \\ e_z < 0 \end{Bmatrix}$$

Two of the planes are always tangent to the light cone, and the other two planes are time-like.

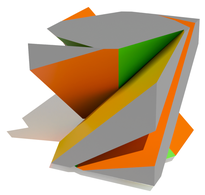


Figure 3:
*The intersection of four crooked half-spaces is a **fundamental domain** for an action determined by two elements in* $\mathrm{SO}(2,1) \ltimes \mathbb{R}^3$.

Definition (Proper action)**.** An action of a discrete group Γ on a topological space X is called ***proper*** if for every compact set $K \subset X$, only finitely many elements $g \in \Gamma$ satisfy

$$gK \cap K \neq \emptyset.$$

Definition (Fundamental domain)**.** A closed subset $F \subset X$ with non-empty interior is a ***fundamental domain*** for a proper action of a discrete group Γ on X if

$$\Gamma \cdot F = \bigcup_{g \in \Gamma} gF = X,$$

and the only element $g \in G$ such that

$$g \operatorname{int} F \cap \operatorname{int} F \neq \emptyset$$

is the identity element.

HYPERBOLIC ELEMENTS AND THE MARGULIS INVARIANT

Definition (Margulis invariant,[2])**.** For $(C, w) \in \mathrm{SO}(2,1) \ltimes \mathbb{R}^3$ with C hyperbolic, the ***Margulis invariant*** is

$$\alpha(C, w) = \langle x^0(C), w \rangle_{2,1}.$$

Intuitively, it's measuring translation along the invariant axis of $x^0(C)$.

We also note that in this setting, $\alpha(C, w) = 0$ is equivalent to (C, w) having a fixed point.

Theorem (Properness Criterion, [3])**.** Let

$$\Gamma = \langle (A_1, v_1), \ldots, (A_k, v_k) \rangle$$

be a subgroup of $\mathrm{SO}(2,1) \ltimes \mathbb{R}^3$, where each A_i is hyperbolic. Then Γ acts properly on $\mathbb{R}^3$ if and only if

$$\inf_{\gamma \in \Gamma} \alpha(\gamma) > 0 \qquad \text{or} \qquad \sup_{\gamma \in \Gamma} \alpha(\gamma) < 0.$$

Meaning all elements have the same sign Margulis invariant and they do not tend to 0.

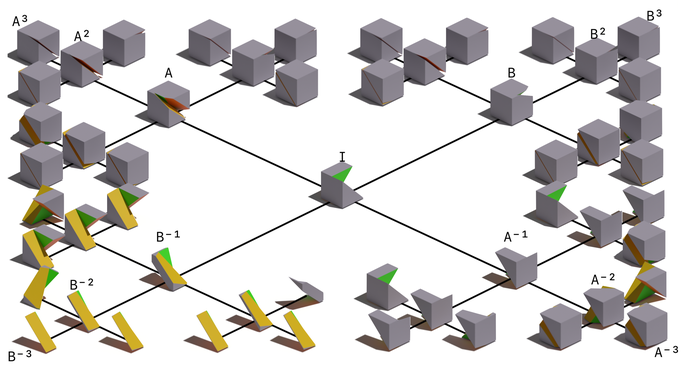


Figure 4: *Translations of a crooked half-space under a proper free group generated by two $SO(2,1) \ltimes \mathbb{R}^3$ elements, arranged as a Cayley graph with depth 3.*

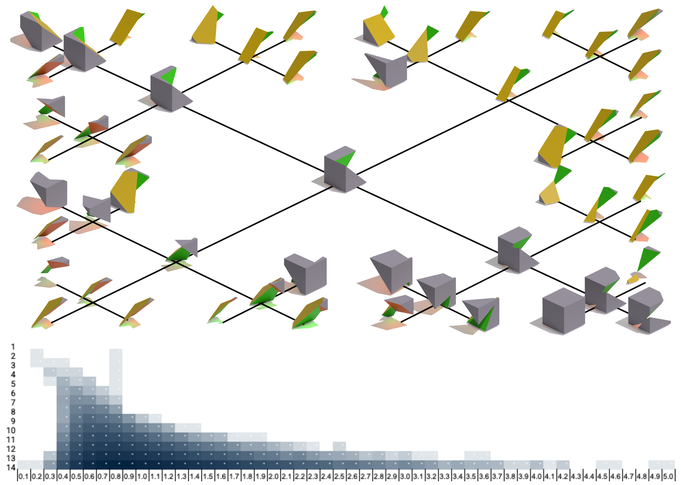


Figure 5: *This histogram shows that the above two actions (see above) have no negative Margulis invariants up to a depth of 15.*

CONCLUSIONS, CONJECTURES, AND OPEN QUESTIONS

In the case when $\Gamma = \langle (A,u), (B,v) \rangle \leq \mathrm{SO}(2,1) \ltimes \mathbb{R}^3$ is such that the normalized Margulis invariant changes sign, we conjecture that there is still a maximal domain $\Omega \subseteq \mathbb{R}^3$ such that the action of Γ on Ω is proper. We formulate the following conjectures and questions about Ω in this case :

Conjecture A (Maximal Domain). We conjecture that no affine plane is contained in Ω unless it is invariant under the entire group Γ.

Conjecture B (Geometry of the Domain). The fundamental domain F retains the structure of a **crooked polyhedron**, bounded by crooked planes determined by the generators of Γ.

Open Question. Since the choice of F is not unique, does there exist a sequence of domains $\{F_n\}$ such that

$$\lim_{n \to \infty} \Omega_n = \lim_{n \to \infty} \Gamma \cdot F_n = \mathbb{R}^3 \, ?$$

We recognize that this research may have applications to general relativity, special relativity, and computational geometry— namely Voronoi diagrams (2D Euclidean Dirichlet Domains). Such applications deserve further exploration.

Références

[1] M. Kapovich and B. Leeb *Domains of discontinuity of Lorentzian affine group actions.* Geom.Dedicata 21891, 2024.

[2] W. M. Goldman. *The Margulis invariant of isometric actions on Minkowski (2+1)-space.* Burger, M., Iozzi, A. (eds) Rigidity in Dynamics and Geometry, Springer, (187–201) 2002.

[3] W. Goldman and F. Labourie and G. Margulis. *Proper affine actions and geodesic flows of hyperbolic surfaces.* Annals of Mathematics 170(1051-1081). 2004.

[4] F. Bonahon. Low-dimensional geometry : from Euclidean surfaces to hyperbolic knots. American Mathematical Soc., 2009

*All visuals are our own. Created in Blender BPY Python Library

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