Optimal Stopping Problems with Applications to Finance

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LOG(M)

ABSTRACT

Optimal stopping involves determining the ideal moment to take an action that maximizes expected rewards. In finance, this concept helps answer questions like:

- When should an investor exercise an American option?
- What's the optimal time to sell an asset?
- How to value flexibility in investment decisions?

THEORETICAL BACKGROUND

Value Equation

Given an asset whose price is determined by a discrete-time stochastic process $(S_t)_{t=0}^T$ we define the following stochastic process, which tracks the value of our asset at each discrete time step $t \in [0, T]$.

The value $V(0, s)$ of an asset with payoff $\Phi(s)$ is:

$$V_n := \sup_{\tau \in [n,T]} \mathbb{E}^{\mathbb{Q}} \left[e^{-r\tau} \Phi(S_\tau) \mid S_n \right]$$

where:

- τ: Exercise time (stopping decision)
- $\mathbb{Q}$: Risk-neutral probability measure
- S_t: Underlying asset price at time t
- r: Risk-free interest rate

We can equivalently define this value process recursively using the Dynamic Programming Equation as follows.

How Does an Investor Formulate the Mathematical Model?

Definition. (Dynamic Programming Equation, [3]) *The option value satisfies:*

$$V_N = e^{-rT} \Phi(S_N)$$
$$V_n = \max \left\{ \mathbb{E}[V_{n+1} \mid \mathcal{F}_n], \Phi(S_n) \right\}, \quad 0 \le n < N$$

This recursive relationship allows backward computation of optimal decisions, comparing immediate exercise value ($\Phi(S_n)$) with expected continuation value.

REFERENCES

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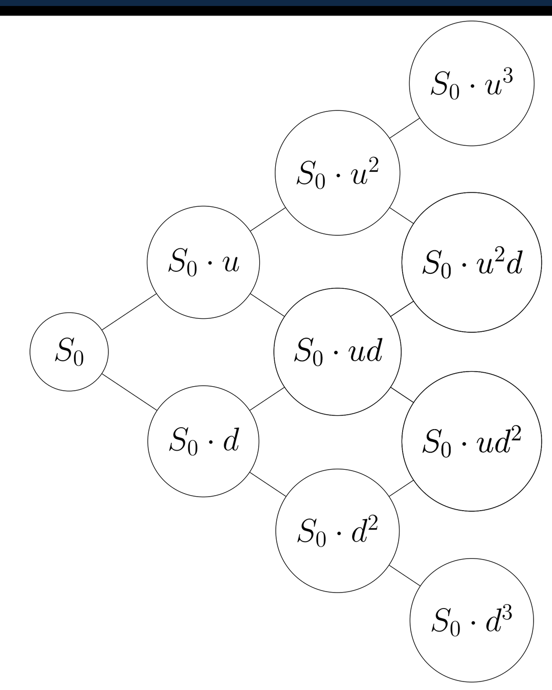
DISCRETE MODELS & SOLUTIONS



Binomial Tree Model [4]

Least-Squares Monte Carlo (LSM) model leverages Monte Carlo simulation to generate multiple paths of the underlying asset price S_t.
Valuing American Options: It estimates the optimal exercise strategy by regressing discounted future cash flows on a set of basis functions of current state variables.

The LSM model proceeds in two main steps:

1. **Path Simulation**:
The underlying asset price paths are generated using the Geometric Brownian Motion (GBM) model [2]. The price at time t is given by:

$$S_t = S_{t-1} \exp\left(\left(r - \frac{1}{2}\sigma^2 \right) \Delta t + \sigma \sqrt{\Delta t}\, Z \right)$$

where r is the risk-free rate, σ is the volatility, Δt is the time step, and Z is a standard normal random variable. This generates M paths of the asset price over N time steps.

2. **Backward Induction**:
Starting from the final time T, the algorithm works backward to determine the optimal exercise strategy at each time step. At expiration $t = T$, set $V_T^{(i)} = \max(K - S_T^{(i)}, 0)$.
For $t = T-1, \dots, 1$: We first identify in-the-money paths $\mathcal{I}_t = \{i : S_t^{(i)} < K\}$; Then We fit regression models (can use different degrees). Specifically, we regress discounted future values on basis functions, here's an example of linear regression:

$$e^{-r\Delta t} V_{t+1}^{(i)} = \sum_{k=1}^{L} \beta_k \phi_k(S_t^{(i)}) + \epsilon^{(i)}, \quad i \in \mathcal{I}_t$$

Then we compute continuation values $\widehat{C}_t^{(i)} = \sum_{k=1}^{L} \widehat{\beta}_k \phi_k(S_t^{(i)})$ and update option values:

$$V_t^{(i)} = \begin{cases} \max(K - S_t^{(i)}, \widehat{C}_t^{(i)}) & i \in \mathcal{I}_t \\ e^{-r\Delta t} V_{t+1}^{(i)} & \text{otherwise} \end{cases}$$

This gives the initial option value:

$$V_0 = \frac{1}{M} \sum_{i=1}^{M} e^{-r\tau_i} V_{\tau_i}^{(i)}$$

where τ_i denotes the optimal exercise time for path i.

Continuous Models

Get to know Brownian Motion

Definition. (Brownian Motion, [5]) *A continuous-time stochastic process W_t satisfies the following properties:*

1. ***Continuity**: $t \mapsto W_t(\omega)$ are continuous almost surely.*
2. ***Independent Increments**: For $s < t$, $W_t - W_s \sim \mathcal{N}(0, t-s)$ and independent of $\mathcal{F}_s$.*
3. ***Initial Condition**: $W_0 = 0$.*

Brownian motion is often termed the "random walk" in continuous time. Its erratic path behavior makes it a natural model for stock price fluctuations.

Black-Scholes Model

Under the risk-neutral measure $\mathbb{Q}$, the stock price S_t follows:

$$dS_t = rS_t\, dt + \sigma S_t\, dW_t,$$

where r is the risk-free rate and σ is volatility. The closed-form solution is:

$$S_t = S_0 \exp\left(\left(r - \frac{\sigma^2}{2} \right) t + \sigma W_t \right).$$

Feynman-Kac and Free-Boundary Problems

Feynman-Kac links PDEs with expectations over stochastic processes. If X_t is a diffusion and $V(T, x) = \Phi(x)$, then

$$V(t, x) = \mathbb{E}^{\mathbb{Q}} \left[e^{-r(T-t)} \Phi(X_T) \mid X_t = x \right],$$

solves the PDE:

$$\frac{\partial V}{\partial t} + \mathcal{L}V - rV = 0.$$

In American option pricing, this yields a **free-boundary problem** with an unknown exercise threshold b, where:

$$\begin{cases} \mathcal{L}V - rV = 0 & \text{for } x < b \\ V(x) = \Phi(x), \quad V'(b^-) = \Phi'(b^+) & \text{(smooth fit)} \end{cases}$$

RESULTS

S	σ	T	Finite Difference American	Linear Regr.	Binomial Regr.	Deg. 3 Hermite Regr.
36	0.2	1	4.478	4.408	4.463	4.464
36	0.2	2	4.84	4.74	4.816	4.827
36	0.4	1	7.101	6.99	7.052	7.059
36	0.4	2	8.508	8.35	8.455	8.482
38	0.2	1	3.25	3.175	3.226	3.231
38	0.2	2	3.745	3.65	3.728	3.74
38	0.4	1	6.148	6.036	6.091	6.092
38	0.4	2	7.767	7.518	7.635	7.629
40	0.2	1	2.314	2.253	2.291	2.293
40	0.2	2	2.885	2.812	2.859	2.882
40	0.4	1	5.312	5.217	5.264	5.265
40	0.4	2	6.92	6.793	6.877	6.901
42	0.2	1	1.617	1.578	1.594	1.595
42	0.2	2	2.212	2.148	2.191	2.201
42	0.4	1	4.582	4.494	4.523	4.529
42	0.4	2	6.248	6.128	6.208	6.219
44	0.2	1	1.11	1.069	1.088	1.107
44	0.2	2	1.69	1.647	1.671	1.686
44	0.4	1	3.948	3.856	3.901	3.89
44	0.4	2	5.647	5.525	5.601	5.614

Generated using 100,000 paths, 50 exercise points per time step, and risk-free interest rate of 6%.Finite Difference values from [1].