Wave Instability: The Calm Be*Fourier* the Storm

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LOG(M)

MOTIVATION

- When tossing a rigid body with unequal dimensions in the air (say a book), there are three convenient axes of rotation, namely the x, y and z axes.
- Two axes exhibit stable behavior; when the book is tossed, it continues to rotate along said axis.
- But, the third axis actually rotates through all three axes, exhibiting unstable rotational behavior.

OBJECTIVE:
We are interested in extending our system from rigid body rotation to three-wave interactions as well as looking at the field of dynamic turbulence.

DEFINITIONS

Three-wave interactions (aka triad interaction) is a system of nonlinear differential equations where each equation relates three unknown functions (or variables).

Modular instability refers to the phenomenon in which one wave in a system is initially unstable resulting in a change in amplitude of other waves in the system [5].

Definition (Three-Wave Dynamical System). *The system of nonlinear differential equations that is of the form*

$$\begin{cases} \dot{B}_1 = -ZB_2B_3 \\ \dot{B}_2 = ZB_1B_3^* \\ \dot{B}_3 = ZB_1B_2^* \end{cases} . \tag{0.1}$$

where Z is the interaction coefficient and each B_i and B_i^* is the amplitude and complex conjugate of the given wave, respectively.

Triad Clusters

- The time evolution of the each amplitude, $\dot{B}_i$, depends on the two connected amplitudes.
- When there are exactly three connected modes, we have a single triad. If we have multiple triads connected, the shared mode's equation would look something like $ZB_1B_2 + ZB_3B_4$, where B_1 and B_2 are in one triad, and B_3 and B_4 are in another.

How do we determine triads?

- Each mode is wave number and frequency pair, $(k_i, \omega(k_i))$.
- In each triad, the mode with the highest frequency is the unstable one, called *A-mode*, and the stable modes are called *P-modes* [1].
- The three wave numbers and frequencies must satisfy the resonance conditions listed below.

Resonance conditions are the set of equations that must be satisfied to create a triad of resonance [2]. The equations are:

$$\begin{cases} k_1 + k_2 = k_3 \\ \omega(k_1) + \omega(k_2) = \omega(k_3) \end{cases} \tag{0.2}$$

TRIAD CLUSTERS

Cluster Structure

- Clusters can be structured in a number of different ways.
- For example, a butterfly cluster: a cluster of triads where each triad is connected by one shared wave mode.
- Over the course of this project we wrote code to numerically model clusters and, in turn, utilize these models to output data using various graphical functions to construct graphs of these clusters.

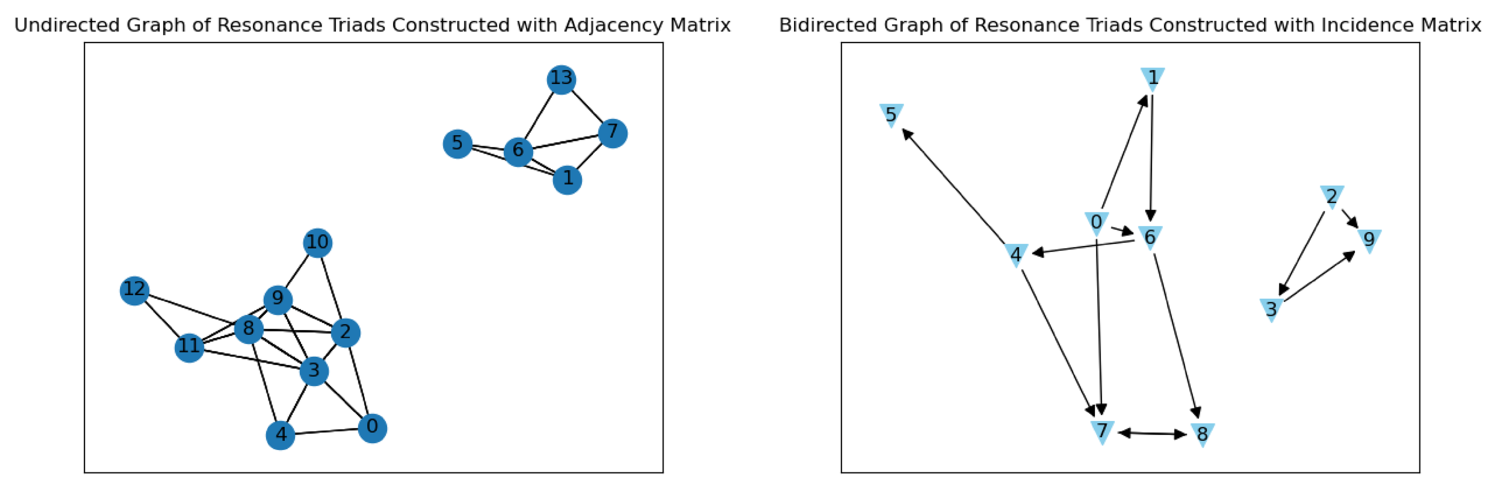

Figure 1: *Graphical representations of clusters using differing graphical methods. Left utilizes an adjacency matrix and right utilizes an incidence matrix.*

MULTIPLE TRIADS

How do we analyze a system of many triads?

- When we have many connected triads, it is called a *resonance cluster.*
- For some wave types that have three-wave interaction, we will have a infinite number of triads which satisfy the conditions. Because we created a numerical simulation of these systems, we had to produce a finite and readable number of triads and cluster.
- The system takes in a *dispersion relation*, which is a function relating ω and k for a specific type of wave.
  - ex. Edge Waves: $\omega(k) = \sqrt{(2n+1)|k|}$, where n is a natural number.
- We aimed to study how energy flows through the cluster, which we call a **cascade**.

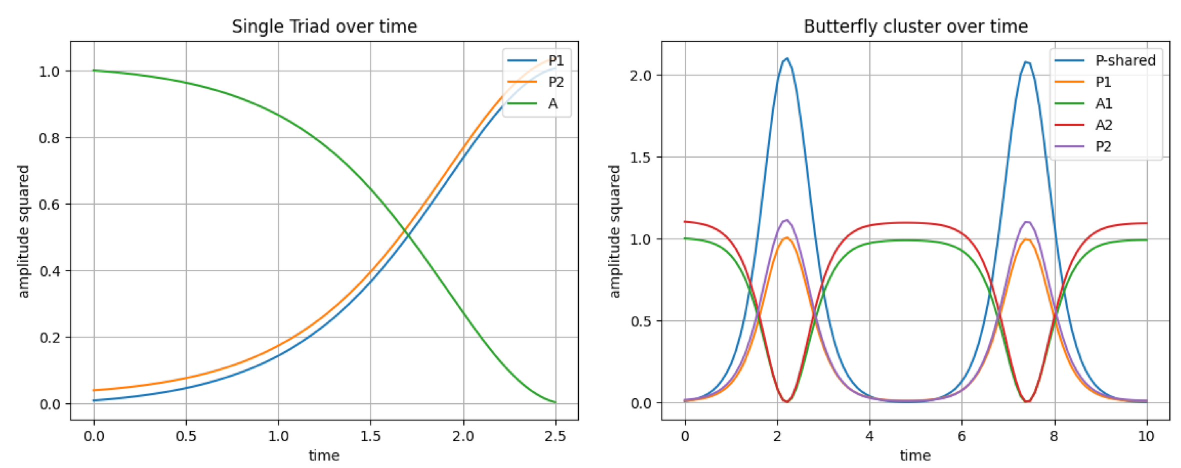

Figure 2: *In the above plots, we see the energy cascade over time in two connected triads with a shared P-mode, and a single triad. The energy starts in the respective A-modes, and transfers into the other modes. We see the recurrent nature of the system when we have no external forces, referred to as Manley-Rowe Invariance [3].*

EXTERNAL FORCES

What happens when we have dissipation on a system?

- One might expect that the equilibrium solution of a system in the presence of damping would be where each mode goes to 0.
- When we add a linear damping term on a strictly P-mode in a system, we see that the system reaches a non-zero equilibrium solution when the energy begins in an A-mode. Modes which are strictly P-modes reach non-zero constants as time increases.
- In an N-chain, we have N-triads each connected to the next by one shared mode. Each left mode in a triad in the A-mode.
- The system looks like:

$$\begin{cases} \dot{B}_1 = -ZB_2B_2 \\ \dot{B}_2 = ZB_1B_3 \\ \dot{B}_3 = ZB_1B_2 - ZB_4B_5 \\ \dot{B}_4 = ZB_3B_5 \\ \dot{B}_5 = ZB_3B_4 - B_5 \end{cases}$$

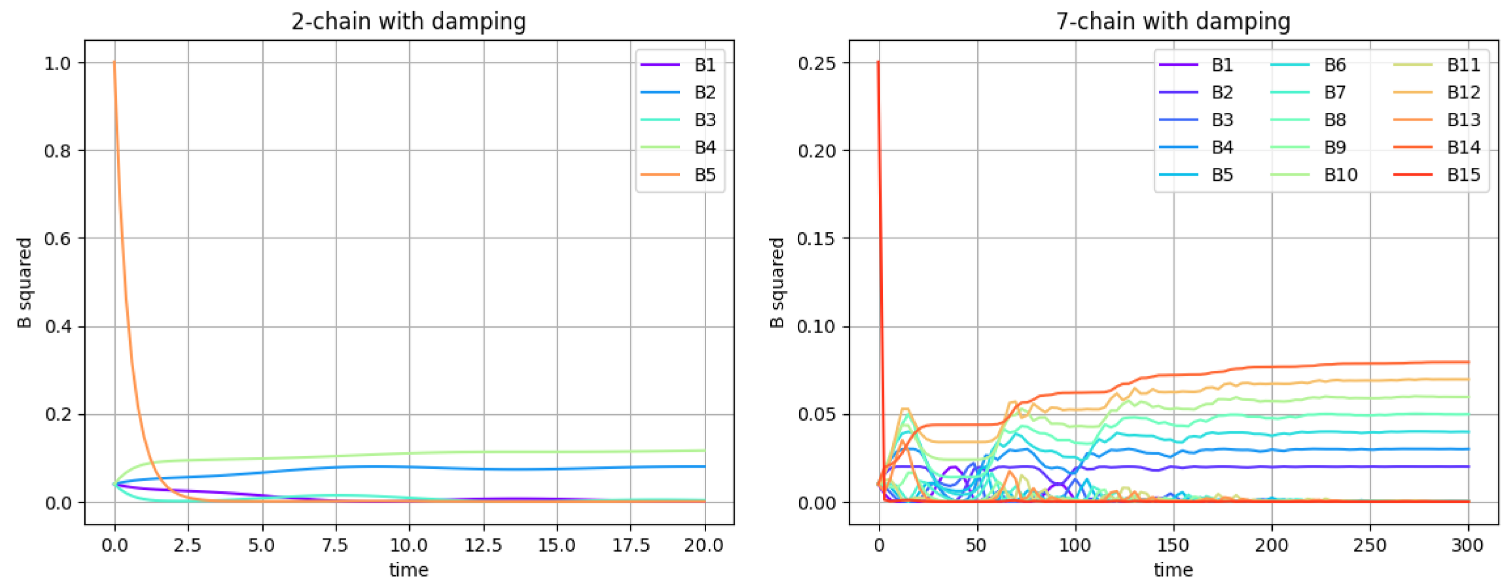

Figure 3: *The figure demonstrates the equilibrium solution to the system above. We see that the sole P-modes of the system reach non-zero equilibrium solutions, whereas all other modes go to 0 as time evolves, for 2 triads and 7 triads.*

FUTURE WORK

- We can generalize the damping results to any N-chain. The solely P-modes are the ones that reach a non-zero equilibrium solution as $t \to \infty$. These results were also proven analytically through solving for the eigenvalues of the Jacobian of the system to determine the stability of each mode.
- For further research, we will also want to study how forcing affects the system, although this will be more difficult to generalize, as the type of wave and specific cluster changes the effect [4].

REFERENCES

[1] Elena Kartashova. *Nonlinear resonance analysis: theory, computation, applications.* Cambridge University Press, New York, 2011.

[2] Elena Kartashova and Miguel D. Bustamante. Resonance clustering in wave turbulent regimes: Integrable dynamics, February 2010. arXiv:1002.4994 [nlin].

[3] Elena Kartashova and Loredana Tec. A constructive method for computing generalized Manley–Rowe constants of motion. *Communications in Computational Physics*, 14(4):1094–1102, October 2013. arXiv:1212.4295 [nlin, physics:physics].

[4] Harvey Segur and Nicolas Grisouard. Triad (or 3-wave) resonances. June 2009.

[5] V. E. Zakharov and L. A. Ostrovsky. Modulation instability: The beginning. *Physica D: Nonlinear Phenomena*, 238(5):540–548, March 2009.

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