RINGS OF INVARIANTS

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Let $\mathrm{GL}_n(\mathbf{C})$ be the group of $n \times n$ invertible matrices over the field $\mathbf{C}$ of complex numbers, and let $\mathbf{C}[\mathrm{GL}_n]$ denote the $\mathbf{C}$-algebra of polynomial functions $\mathrm{GL}_n(\mathbf{C}) \to \mathbf{C}$; this is isomorphic to $\mathbf{C}[\{X_{ij}\}_{1 \leq i,j \leq n}][1/\det]$, where det is the determinant function (a polynomial!). There is a natural action of $\mathrm{GL}_n(\mathbf{C})$ on $\mathbf{C}[\mathrm{GL}_n]$ by $(g \cdot f)(x) = f(g^{-1}xg)$. It is natural to ask for the fixed points of this action, the so-called *class functions*; the set of these is denoted by $\mathbf{C}[\mathrm{GL}_n]^{\mathrm{GL}_n}$. (This superscript is generally used to mean that we are taking fixed points.) Because any class function f depends only on the conjugacy class of its input, a relatively short argument shows that it only depends on the *characteristic polynomial* of its input. Arguing along these lines, one shows that $\mathbf{C}[\mathrm{GL}_n]^{\mathrm{GL}_n}$ is isomorphic to the set of polynomial functions on the set X of possible characteristic polynomials. A characteristic polynomial is determined by its coefficients, which for an invertible matrix can be arbitrary (except for the constant term, which cannot be 0). A polynomial function on X just depends on these coefficients, so the ring of such functions is isomorphic to $\mathbf{C}[\{X_i\}_{1 \leq i \leq n-1}, X_0^{\pm 1}]$ (almost, but not quite, a polynomial ring). Whereas $\mathbf{C}[\mathrm{GL}_n]^{\mathrm{GL}_n}$ might look rather daunting, this shows that it is remarkably simple after all!

What happens if $\mathrm{GL}_n(\mathbf{C})$ is replaced by another group? Let $\mathrm{PGL}_n(\mathbf{C})$ be the quotient of $\mathrm{GL}_n(\mathbf{C})$ by the subgroup of scalar matrices. In other words, an element of $\mathrm{PGL}_n(\mathbf{C})$ is just an $n \times n$ complex matrix, considered up to $\mathbf{C}^{\times}$-scaling. We can ask the same question as the previous paragraph: how can one describe the $\mathbf{C}$-algebra $\mathbf{C}[\mathrm{PGL}_n]^{\mathrm{PGL}_n}$ of class functions? If $n = 2$, one can again use considerations with characteristic polynomials to compute that $\mathbf{C}[\mathrm{PGL}_2]^{\mathrm{PGL}_2} \cong \mathbf{C}[X]$ is a polynomial ring. However, if $n = 3$, it turns out that this is no longer the case! In fact, it is surprisingly difficult to describe $\mathbf{C}[\mathrm{PGL}_3]^{\mathrm{PGL}_3}$ in a way that elucidates many of its properties.

In this project, we aim to:

- Understand the description of $\mathbf{C}[\mathrm{GL}_n]^{\mathrm{GL}_n}$ given above, and ideally to understand the geometry of the inclusion $\mathbf{C}[\mathrm{GL}_n]^{\mathrm{GL}_n} \to \mathbf{C}[\mathrm{GL}_n]$.
- Compute $\mathbf{C}[\mathrm{PGL}_n]^{\mathrm{PGL}_n}$ for small n (at least $n = 3$ and 4), and try to understand its ring-theoretic properties.
- Compute various other algebraic quantities related to the inclusion $\mathbf{C}[\mathrm{PGL}_n]^{\mathrm{PGL}_n} \subset \mathbf{C}[\mathrm{PGL}_n]$.

Time permitting (and depending on the interests of the participants), we might try to understand related groups of matrices such as the projective symplectic groups PSp_{2n}.

Some experience in linear and abstract algebra is important to work on this project, but I believe that there are no other prerequisites.