Ring of Invariants

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ABSTRACT

Invariant theory explores polynomial functions that remain unchanged under a group action. We focus on the fixed subrings of polynomial algebras under conjugation by the general linear group $GL_n(\mathbb{C})$ and its projective quotient $PGL_n(\mathbb{C})$. Our goals are:

- Identify explicit generators and relations for small n.
- Connect combinatorial descriptions of weight semigroups to algebraic invariants.
- Demonstrate computational methods for verifying Hilbert bases.

OBJECTIVE: Provide a clear, visual guide to the construction and structure of $C[PGL_n]^{PGL_n}$ for $n = 1, 2, 3$.

BACKGROUND

Invariant Theory and Matrix Groups

Invariant theory studies functions unchanged under group actions. In our case, the group is the general linear group:

- $\mathrm{GL}_n(\mathbb{C})$: all invertible $n \times n$ complex matrices[1].
- $\mathrm{PGL}_n(\mathbb{C}) = \mathrm{GL}_n(\mathbb{C})/\mathbb{C}^\times$: projective group modulo scalar matrices[2].

Invariant Rings:
The ring $\mathbb{C}[\mathrm{GL}_n]$ is the algebra of polynomial functions on $\mathrm{GL}_n(\mathbb{C})$, and we define[3]:

$$\mathbb{C}[\mathrm{GL}_n]^{\mathrm{GL}_n} = \{f \in \mathbb{C}[\mathrm{GL}_n] \mid f(g^{-1}xg) = f(x)\}$$

These are "class functions", depending only on eigenvalues. Similarly for PGL_n, we study $\mathbb{C}[\mathrm{PGL}_n]^{\mathrm{PGL}_n}$.

$$\mathbb{C}[PGL_n]^{PGL_n} \cong \mathbb{C}[T^n]^{S_n} \cong \mathbb{C}[V_n]^{S_n},$$

where

$$V_n = \{(x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n \mid x_1 + x_2 + \cdots + x_n = 0\}.$$

The symmetric group S_n acts on V_n by permuting the coordinates.

$\mathbb{C}$-Algebra
Let V be an abelian group. The complex group algebra $C[V]$ is defined [4] as the $\mathbb{C}$-vector space with basis elements $\{t^v \mid v \in V\}$. The multiplication on $C[V]$ is given by linear extension of the rule

$$t^v \cdot t^w = t^{v+w}, \quad \forall v, w \in V.$$

Since V is abelian, the multiplication is commutative, and thus $C[V]$ is a commutative C-algebra.

Example: The C-Algebra of $\mathbb{Z}$ and $\mathbb{Z}^n$

For $C[\mathbb{Z}]$:

$C[\mathbb{Z}]$ is the group algebra of the group $\mathbb{Z}$. Since every element of $\mathbb{Z}$ can be written as an integer n, we denote the corresponding basis element by t^n. With the multiplication rule

$$t^n \cdot t^m = t^{n+m},$$

$C[\mathbb{Z}]$ becomes isomorphic to the ring of Laurent polynomials $C[t, t^{-1}]$.

For $C[\mathbb{Z}^n]$:

In a similar way, every element of $\mathbb{Z}^n$ is an n-tuple $(v_1, v_2, \ldots, v_n)$. We denote the corresponding basis element by t^v, which can also be written as

$$t^v = t_1^{v_1} t_2^{v_2} \cdots t_n^{v_n}.$$

The multiplication rule is

$$t^v \cdot t^w = t^{v+w},$$

where the addition is done component-wise. Therefore,

$$C[\mathbb{Z}^n] \cong C[t_1, t_1^{-1}, t_2, t_2^{-1}, \ldots, t_n, t_n^{-1}].$$

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MAIN THEORY

Theorem 1. *We have the following isomorphisms[3]:*

$$C[\ PGL_n]^{\ PGL_n} \cong C[T^n]^{S_n} \cong C[V_n]^{S_n},$$

where

$$V_n = \{(x_1, x_2, \ldots, x_n) \in Z^n \mid x_1 + x_2 + \cdots + x_n = 0\}.$$

The symmetric group S_n acts on V_n by permuting the coordinates.

Theorem 2. *Let $\lambda = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n$ be a weight for $SL_n(\mathbb{C})$ satisfying*

$$x_1 + x_2 + \cdots + x_n = 0.$$

Then λ is dominant if and only if

$$x_1 \geq x_2 \geq \cdots \geq x_n.$$

Cases $n = 1$ and $n = 2$

Case $n = 1$:
For $n = 1$, we have

$$V_1 = \{0\},$$

so the torus is $T^1 \cong \mathbb{C}^\times$. Hence,

$$C[T^1]^{S_1} \cong C[t, t^{-1}],$$

which is simply the full Laurent polynomial ring.

Case $n = 2$:
For $n = 2$, the weight space is

$$V_2 = \{(x, -x) \mid x \in \mathbb{Z}\}.$$

Each weight $(x, -x)$ is assigned the basis element $t^{(x,-x)}$ in the group algebra $C[V_2]$. In particular, define

$$u = t^{(1,-1)}.$$

The symmetric group S_2 acts on V_2 by interchanging the two components. In order to construct an S_2-invariant, we consider

$$u + u^{-1}.$$

Moreover, one can show (by induction, for example) that for any positive integer n

$$u^n + u^{-n}$$

can be expressed as a polynomial in $u + u^{-1}$ (e.g., $u^2 + u^{-2} = (u + u^{-1})^2 - 2$, etc.). Thus, we obtain

$$C[V_2]^{S_2} \cong C[u + u^{-1}].$$

Case $n = 3$

For $n = 3$, after an appropriate linear transformation, the semigroup of dominant weights can be described by

$$D_3 = \{(m, n) \in \mathbb{Z}^2_{\geq 0} \mid 2m \geq n,\ 2n \geq m\}.$$

By observation, we conjecture that the minimal generators for D_3 are

$$(1, 1), \quad (2, 1), \quad (1, 2).$$

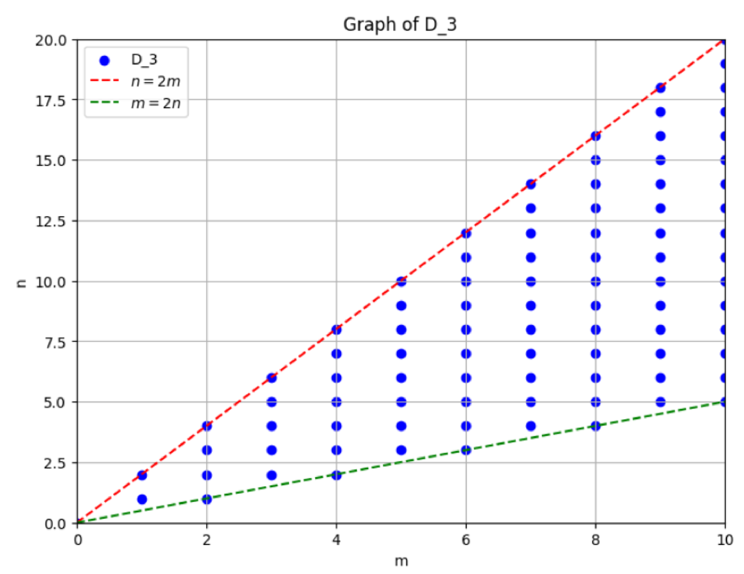

Figure 1: *Visualization of D3*

We can use induction to show that every point $(m, n) \in D_3$ can be expressed as a non-negative integer combination of these generators. First, we construct our invariants by enforcing the condition that the sum of the coordinates is zero. We set

$$(a, b, c) = (m, -n, n - m),$$

so that

$$a + b + c = 0.$$

Thus, the corresponding three sigma invariants are given by

$$\sigma_{(1,0,-1)}, \quad \sigma_{(1,1,-2)}, \quad \sigma_{(2,-1,-1)}.$$

Each invariant is defined as a sum over the S_3-orbit of a basic generator. For example, we define

$$\sigma_{(1,0,-1)} = t^{(1,0,-1)} + t^{(1,-1,0)} + t^{(-1,1,0)} + t^{(-1,0,1)} + t^{(0,1,-1)} + t^{(0,-1,1)},$$

where $t^{(x,y,z)}$ denotes the basic generator corresponding to the weight (x, y, z). This construction ensures that $\sigma_{(1,0,-1)}$ is invariant under the action of S_3.
Similarly, $\sigma_{(1,1,-2)}$ and $\sigma_{(2,-1,-1)}$ are constructed in the analogous way (their sums involve only three distinct terms since two coordinates coincide).
Using these definitions, one can derive the following relation:

$$\begin{aligned}\sigma^3_{(1,0,-1)} &= 9 + 9\,\sigma_{(1,0,-1)} + 6\,\sigma_{(2,-1,-1)} + 6\,\sigma_{(1,1,-2)} \\ &\quad + 3\,\sigma_{(1,0,-1)}\,\sigma_{(2,-1,-1)} + 3\,\sigma_{(1,0,-1)}\,\sigma_{(1,1,-2)} \\ &\quad + \sigma_{(2,-1,-1)}\,\sigma_{(1,1,-2)}\end{aligned}$$

This relation encapsulates the interplay between the sigma invariants constructed from the basic generator t and serves as the final result in our derivation.

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