Realizing Transfer System

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ABSTRACT

In a saturated transfer system, arrows are drawn when everything in S_n comes from exactly p things from S_d, where np = d. Our research goals are to find the necessary and sufficient conditions for a saturated transfer system $\mathcal{R}$ to be realizable, in other words, $\mathcal{R} = Lin(S)$ for some S; and find the necessary and sufficient condition for a compatible pair $(\mathcal{R}, \mathcal{K})$ to be realizable.

The motivation for this project arises from exploring the intricate algebraic structures within homotopy theory, particularly through the lens of transfer systems, which are essential for understanding interactions among divisors of a given integer n.

THEORETICAL BACKGROUND

Definition (Fact(n))**.** *Given a positive integer n,* $\mathrm{Fact}(n)$ *is defined as a set of positive factors of n.*

- *For example, if* $n = 18$*, then* $\mathrm{Fact}(n) = \{1, 2, 3, 6, 9, 18\}$.

Definition (complete transfer system)**.** *A graph with given positive integer n. Its vertices* $\mathrm{Fact}(n)$ *and edges between all pairs* $d \to pd$ *such that* p *is a prime and* $p, pd \in \mathrm{Fact}(n)$.

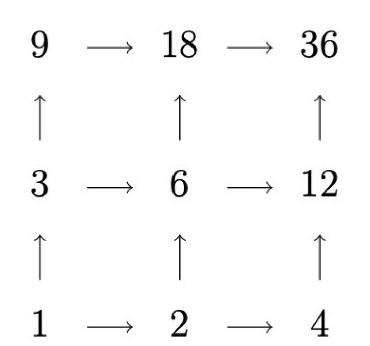


Figure 1: *Complete Transfer System with n = 36*

Definition (Z/n)**.** *Let* $a, n \in Z$, $a = q * n + r$*, then* $a \equiv r$ *(mod* n*) in* Z/n.

$$\mathbb{Z}/n = \{[0], [1], [2], \ldots, [n-1]\},$$

where $[a]$ *denotes the equivalence class of* a*, containing all integers* $b \in \mathbb{Z}$ *such that* $b \equiv a \pmod{n}$.

Definition (Lin(S))**.** • *For a subset S of* $\{0, ..., n-1\}$*, the linear isometry transfer system Lin(S) is the graph with vertices in Fact(n), and an edge from* e *to* d *exists iff there is a prime* p*, such that* $d = pe$*, and* $|S_d| = p|S_e|$

Definition (cofibrant)**.** *An element* $d \in \mathrm{Fact}(n)$ *is called* ***cofibrant*** *for a transfer system R if there is no arrow* $\frac{d}{p} \to d$.

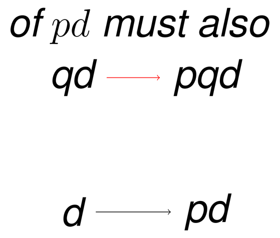
Definition (St(S))**.** • *The Steiner cosaturated transfer system associated to a subset S of* $\{0, ..., n-1\}$ *is*

$$St(S) = \{gcd(n, s_1, s_2..., s_k), k \geq 1, s_i \in S\} \qquad (0.1)$$

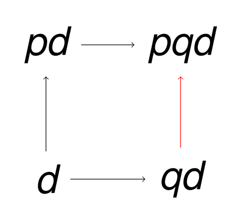
THEOREMS

A **saturated transfer system** on a group G is a partial order $\to$ that satisfies the following theorems:

Theorem 1. *([1]) For any arrow* $pd \to pqd$ *(where* p, q *are primes), every factor* d *of* pd *must also have an arrow* $d \to qd$.



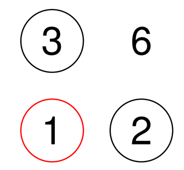
Theorem 2. *([1]) If* $p \neq d$ *(where* p *and* q *are distinct primes) and there are arrows* $d \to pd$ *and* $pd \to pad$*, then by Rule 1, we are guaranteed to have an arrow* $d \to qd$ *and an arrow* $qd \to pqd$.



A **cosaturated transfer system** K on the factor set Fact(n) is a non-empty subset $K \subseteq$ Fact(n) that satisfies the following theorem.

Theorem 3. *([1]) If* d *and* e *are elements of* K*, then* $\gcd(d, e)$ *must also be in* K.

Below is an example of K with $d = 2$ and $e = 3$:

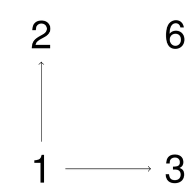


If R is a saturated transfer system and K is a cosaturated transfer system, then we say (R, K) is a **compatible pair** if it also satisfies the following thorem:

Theorem 4. *([2]) For any arrow* $d \to pd$ *of* R*, and any* $e \in K_d$ *such that* $\gcd\left(\frac{d}{e}, p\right) = 1$*, we have both* $e \in K_{pd}$ *and* $pe \in K_{pd}$.

EXAMPLE

For $n = 6$, suppose $S = S_6 = \{1, 2, 3, 5\}$, then $S_3 = \{0, 1, 2\}$, $S_2 = \{0, 1\}$, and $S_1 = \{0\}$. Therefore, the corresponding transfer system for this particular S is pictured as follows:



In turn, $S = \{1, 2, 3, 5\}$ realizes the graph.

RESULTS

We develop a **"remove strategy"** to approach the realization questions. If given R, we always assume the set of S_n is {0, 1, ..., n - 1}, and we first remove numbers that are not in the set of $S_{\frac{n}{p}}$ and the set of $S_{\frac{n}{q}}$, and then remove the last number in S_n.

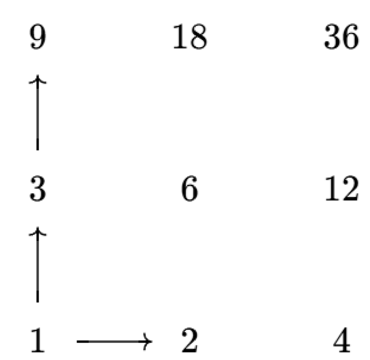


Figure 2: *R in the* $\mathbb{Z}$ */ 36 universe*

In this case, we intuitively know from the arrows that S_1 = {0}, S_2 = {0, 1}, S_3 = {0, 1, 2}, and S_9 = {0, 1, 2, 3, 4, 5, 6, 7, 8}. According to our "remove strategy," we assume S_6 = {0, 1, 2, 3, 4, 5}, and we remove whatever conflicts the realization, which happened to be nothing in this case, so we just remove the last number 5, and get S_6 = {0, 1, 2, 3, 4}. In order to know S_{12}, we first assume S_4 = {0, 1, 2}, then assume S_{12} = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}. Then we remove what is absent in S_4, which is 3, 1*4+3=7, and 2*4+3=11, and remove what is absent in S_6, which is 5, and 1*6+5=11. Finally we get S_{12} = {0, 1, 2, 4, 6, 8, 9, 10}. We repeat these steps for every vertices, and then eventually we get a set for S_{36} that realizes R.

We define $CardSat(G)$ to be the cardinality of saturated transfer system of cyclic group G.

For the case $CardSat(\mathbb{Z}/p_1^m p_2^n)$:

- CardSat($\mathbb{Z}/p_1^n$) = 2^n.
- CardSat($\mathbb{Z}/p_1^n p_2$) = $3^{n+1} - 2^n$

The following recurrence relation holds:

$$CardSat(\mathbb{Z}/p_1^m p_2^n) = \Sigma_{i=0}^{n-1} \binom{n}{i+1} CardSat(\mathbb{Z}/p_1^{m-1} p_2^{n-i})$$

Solving the recurrence we get:

$$CardSat(\mathbb{Z}/p_1^m \, {}_2 2^n) = \sum_{j=2}^{m+2} (-1)^{m-j} \left\{ \begin{matrix} m+1 \\ j-1 \end{matrix} \right\} \frac{j!}{2} j^n$$

FUTURE WORK

Prove/disprove:

Given natural number n, and saturated transfer system R. If 2 is not cofibrant, then $R = Lin(S)$, for some $S \subseteq \{0, 1, 2..., n-1\}$.

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Thank you, Bar Roytman, for supporting our research project.