Spine of Ornamentation Lattices

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ABSTRACT

Ornamentation lattices are a recent generalization of the Tamari lattice. This project aims to develop an efficient algorithm to determine the spine of an ornamentation lattice and analyze the behavior of its join irreducible elements. By doing so, we can devise ways to compute the number of maximum-length chains within these lattices.

THEORETICAL BACKGROUND

Definition (Ornament)**.** *An ornament $\mathfrak{o}$ of T is a connected subgraph of T. Observe that every ornament o has a unique minimal element $v_{\mathfrak{o}}$; we say $\mathfrak{o}$ is hung at $v_{\mathfrak{o}}$.*

Definition (Ornamentation)**.** *Let $Orn(T)$ denote the set of ornaments of T. An ornamentation of T is a function $O : T \to Orn(T)$ such that:*

- *for all $v \in T$, the ornament $\mathfrak{o}(v)$ is hung at v*
- *for all $u, v \in T$, the ornaments $\mathfrak{o}(u)$ and $\mathfrak{o}(v)$ are either nested or disjoint.*

Definition (Ornamentation Lattice)**.** *Let T be a rooted plane tree. Let the partial order $\leq$ be defined such that*

$$\delta \leq \delta' \quad \text{if and only if} \quad \delta(v) \subseteq \delta'(v) \quad \text{for all } v \in T$$

Then the Poset $\mathfrak{O}(T)$ is a lattice whose meet operation $\wedge$ is such that

$$(\delta \wedge \delta')(v) = \delta(v) \cap \delta'(v) \text{for all } v \in T.$$

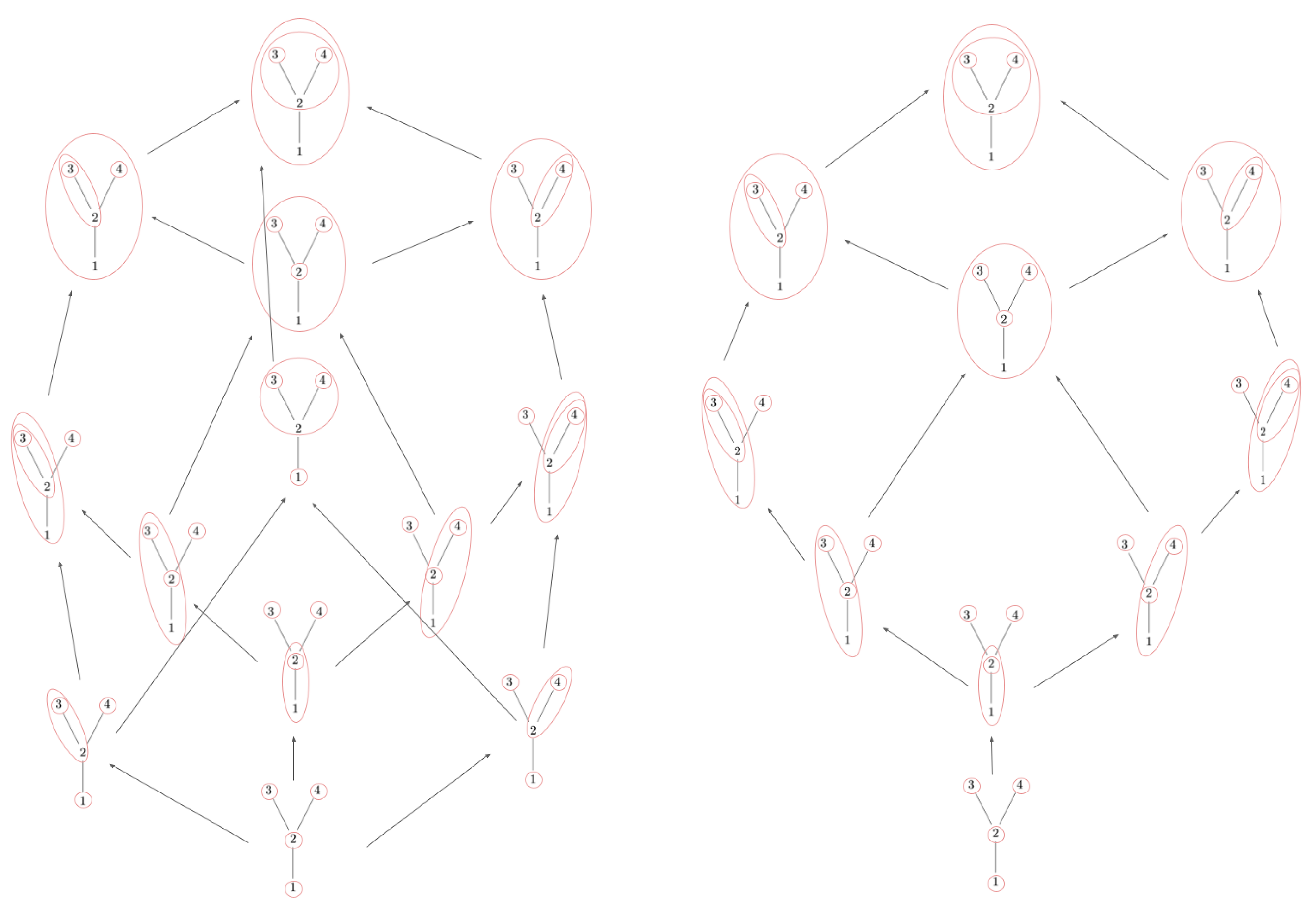

Figure 1: *From left to right: Ornamentation lattice and its spine.*

Definition (Spine)**.** *The spine of $\mathfrak{O}(T)$, denoted Spine($\mathfrak{O}(T)$), is the subposet of elements of $\mathfrak{O}(T)$ that belong to at least one maximal length chain.*

THEOREMS

Theorem 1. *Let L be an ornamentation lattice. Then L is a trim lattice.*

Theorem 2. *Let L be a trim lattice. Then the spine of L, denoted $Spine(L)$, is a distributive lattice.*

Theorem 3. *Let L be a finite distributive lattice. Then the number of linear extensions of the poset of join-irreducible elements of L is equal to the number of maximal chains in L.*

Theorem 4. *Distributive lattice is isomorphic to the poset of order ideals of subposet of join-irreducibles.*

RESULTS

Definition. *Let $T = (V, E)$ be a rooted tree. The function subtreeSize : $V \to \mathbb{N}$ is defined by*

$$\text{subtreeSize}(v) = \text{the number of nodes in the subtree rooted at } v \quad \forall v \in V.$$

Theorem 5. *The length of a maximal-length chain in $(\mathfrak{O}(T), \leq)$ is*

$$\sum_{v \in V} \text{subtreeSize}(v) - |V| + 1 = \sum_{v \in V, v \neq r} \text{subtreeSize}(v) + 1.$$

Theorem 6. *Let $\{\rho_1 \leq, \ldots, \leq \rho_n\}$ be a maximum-length chain of an ornamentation lattice. Then, for all $i \in [2, n] \cap \mathbb{N}$, there exists exactly one vertex v such that $|\rho_i(v)| - |\rho_{i-1}(v)| = 1$. For all vertices $u \neq v$, $|\rho_i(u)| - |\rho_{i-1}|(u) = 0$.*

Definition. (Genetic) *Let $T = (V, E)$ be a rooted tree. Let $\rho \in \mathfrak{O}(T)$. ρ is genetic if for all $v \in V$, for all of the children vertices c of v, $\rho(c) \subseteq \rho(v)$ or $\rho(c) = \{c\}$.*

Theorem 7. *Let $T = (V, E)$ be a rooted tree. Let $\rho \in \mathfrak{O}(T)$. Then ρ is an element of Spine($\mathfrak{O}(T)$) if and only if ρ is genetic.*

Definition (Poset $\mathcal{P}_{\text{pairs}}$)**.** *Let $T = (V, E)$ be a rooted tree with root r. Let $<_T$ denotes the proper ancestor relation (Parent $<_T$ Child). We define the set of ancestor-descendant pairs:*

$$P_{\text{pairs}} = \{(u, v) \in V \times V \mid u <_T v\}.$$

and a partial order $\preceq_P$ on P_{pairs}:

$$(u_1, v_1) \preceq_P (u_2, v_2) \iff u_1 \preceq_T u_2 \text{ and } v_1 \preceq_T v_2.$$

We denote the resulting poset $(P_{\text{pairs}}, \preceq_P)$ as $\mathcal{P}_{\text{pairs}}(T)$.

Theorem 8. *The poset of join-irreducible elements of Spine($\mathfrak{O}(T)$) is isomorphic to the poset $\mathcal{P}_{\text{pairs}}(T)$:*

$$(JI(\mathcal{S}), \preceq) \cong \mathcal{P}_{\text{pairs}}(T).$$

FIGURES

Figure 2: *Left: A non-genetic ornamentation; Right: A genetic ornamentation*

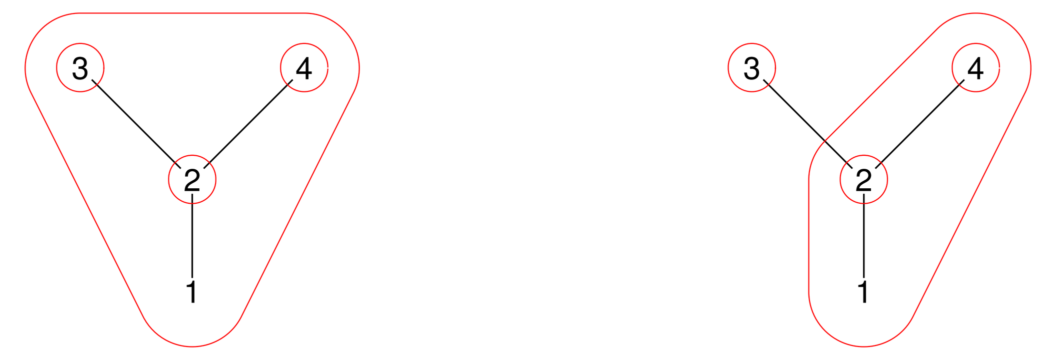

Figure 3: *Left: A Genetic Ornamentation that is not in the Poset of Join Irreducibles of the Spine; Right: An Ornamentation that is in the Poset of Join Irreducibles of the Spine*

FUTURE WORK and APPLICATIONS

- Analyzing patterns of the spine of join-irreducible elements. It's been proved that there is a staircase shape for the spine of join irreducibles for Tamari lattices!
- Ornamentation lattices seem to be relevant in quantum mechanics, particularly Feynman diagrams that can take the structure of rooted trees.
- They prove to be constructions of posets that can be studied independently in other fields such as Cryptography, but that needs further analysis.

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