Large Scale Geometry of Integers

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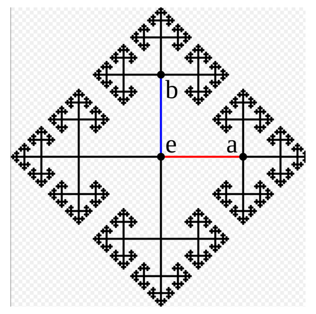
ABSTRACT

- This project focuses on developing algorithms to compute the cheapest way to generate an element of various finitely generated groups
- In particular, we focus on the Integers $\mathbb{Z}$ and the 4x4 Integer Heisenberg Group $H(4)$
- We prove the correctness of our algorithms then implement them in Python

OBJECTIVE: Develop word metric algorithms on the Integer group $\mathbb{Z}$ and the 4-Dimensional Integer Heisenberg group $H(4)$

THEORETICAL BACKGROUND



Definition. *Word Metric :*

- *The **word metric** on a finitely generated group G with generating set X is a function [3]:*

$$|g|_X = \min\left\{n \in \mathbb{N} : g = x_1^{k_1} x_2^{k_2} \dots x_n^{k_n}, x_i \in X\right\}$$

- *The word metric $d(g,h)$ between two elements $g,h \in G$ is defined as:*

$$d(g,h) = |g^{-1}h|$$

WORD METRIC OF THE INTEGER GROUP

- **Objective**: develop an algorithm that computes the word-metric on $\mathbb{Z}$ given generators
- **Inputs**: List of generators ($X = \{x_1, ... x_n\}$), integer to compute word-length of (n)
- **Output**: k, the word-length of n with respect to X
- **Notation**: $|n|_X = k$ if and only if the word-length of n with respect to X is k
- **Note:** we only consider generators that generate $\mathbb{Z}$

Key Lemma:
Suppose X is a finitely generated subset of $\mathbb{Z}$ that generates $\mathbb{Z}$, with $|X| = k$. We define $M = \frac{x_1+x_2}{2}x_1 + \frac{x_2+x_3}{2}x_2 + ... + \frac{x_k+x_{k-1}}{2}x_{k-1} + x_k$, with x_i being the ith term when we order X. Then, for all $n > M$, $|n|_X = |n - x_k|_X + 1$.

- The lemma gives us an efficient way to compute $|n|_X$ using linear recurrence
- We will hence precompute all $|k|_X$ for $k < M$, then use recurrence to compute $|n|_X$

Algorithm for Word-Metric on $\mathbb{Z}$
Given the ordered generating set $X = \{x_1, x_2, \dots, x_k\} \subset \mathbb{Z}$, the word length $|n|_X$ of any integer $n \in \mathbb{Z}$ can be computed as follows:

- Verify that $\gcd(X) = 1$ to ensure X generates $\mathbb{Z}$ (Bezout's Lemma) [4].
- Compute the constant M as defined in the Lemma.
- Precompute the word lengths $|n|_X$ for all integers with $|n| \le M$.
- For $|n|_X > M$, apply the recurrence relation: $|n|_X = |n - x_k|_X + 1$ until we hit a precomputed value. Return the result.

WORD METRIC ON HEISENBERG GROUP

Definition. *4 by 4 Heisenberg group with integer entries: A group with the elements of the form:*

$$\begin{pmatrix} 1 & x_1 & x_2 & z \\ 0 & 1 & 0 & y_1 \\ 0 & 0 & 1 & y_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $x_1, x_2, z, y_1, y_2 \in \mathbb{Z}$ [5].
Denote such Heisenberg group as H(4).

Theorem 1. *The generating elements of $H(4)$ are:*

$$X_1 = \begin{pmatrix} 1&1&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 1&0&1&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&1 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&1 \\ 0&0&0&1 \end{pmatrix}.$$

Any element of the H(4) group can be expressed via the product of any of these matrices.

- Heisenberg groups are important examples of *nilpotent groups*
- Similarly to the Integer word-metric, we make use of some important lemmas to simplify computing word-lengths of Heisenberg matrices

Lemma 1. Partial Commutativity of generating element
These pairs of generating matrices are commutative.

$$X_1X_2 = X_2X_1, \quad X_1Y_1 = Y_1X_1, \quad Y_2X_2 = X_2Y_1, \quad Y_2Y_1 = Y_1Y_2.$$

We are able to express the word of an element $H \in H(4)$, where

$$H = W_1W_2 \cdots W_kW_{k+1} \cdots W_n,$$

as a product of two sequences, where:

$$A = W_1W_2 \cdots W_k, \quad \text{Generated by the pair } X_1, Y_2,$$
$$B = W_{k+1}W_{k+2} \cdots W_n, \quad \text{Generated by the pair } X_2, Y_1.$$

Lemma 2. Group Homomorphism Theorem
The set generated by each pair of commutative matrices is isomorphic to $H(3)$, the 3x3 Heisenberg Group, which has 2 generators

- Blachere ([5]) gives us an analytic formula to solve for the word-length of any 3x3 Heisenberg Matrix. We use this result to divide our 4x4 matrix into two instances of computing word-lengths of 3x3 matrices
- We now just require a way to solve for the 'cheapest' z in the split cases

Algorithm for Word-Metric on $H(4)$ Given a list x_1, x_2, y_1, y_2, z as corresponding to the definition, we compute the word-length of the matrix with these values as follows:

- Extract values of x_1, x_2, y_1, y_2, z
- Derive analytic formulas $L(x_1, y_2, z)$ and $L(x_2, y_1, z)$ with z as a free variable using Blachere's result
- Optimize by solving for $\lfloor \min\left(L(x_1, y_2, z_0) + L(x_2, y_1, (z - z_0))\right) \rfloor$ analytically

The key insight is that we can use case-by-case techniques to calculate the optimal 'split' of the variable z amongst our two generating pairs.

RESULTS

We were successful in proving computational formulas for both groups we considered.

Algorithm Name	Runtime Complexity	Space Complexity
Word-Metric Algorithm for $\mathbb{Z}$	$O(1)$	$O(1)$
Word-Metric Algorithm for Integer $H(4)$	$O(n)$	$O(1)$

The QR code links to our github repository of code

FUTURE WORK

- We want to study the large scale structure of $\mathbb{Z}$ through the lens of the open problem of **Rough Isometry**, which requires the diverse study of metrics on $\mathbb{Z}$
- Word-metrics are examples of **Rough-Isometries**: and we can extract insights from studying them to study the open problem

FUTURE RESEARCH: Make progress on the Rough Isometry of Z problem using insights from these examples

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