Geodesic Flow in Special Linear Group

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Abstract

A geodesic is the shortest curve connecting two point, geodesics define classical geometric object as the sphere in more abtracts geometries. Our project seeks to visualize the geodesics of $\mathrm{SL}_2(\mathbb{R})$ by using Hamiltonian approach. For the main objectives, we will:

1. *Identify the Lie algebra of* $\mathrm{SL}_2(\mathbb{R})$*;*
2. *Endow* $\mathrm{SL}_2(\mathbb{R})$ *with coordinates;*
3. *Determine the geodesic equations;*
4. *Visualize the time evolution of the geodesics.*

$\mathrm{SL}_2(\mathbb{R})$ as a group

Definition. *The* ***special linear group of*** 2×2 ***matrices****, is the set of* 2×2 *matrices of determinant 1.*

- Closure: The expression $\det(AB) = \det(A)\det(B)$ guarantees that if A and B are in $\mathrm{SL}_2(\mathbb{R})$ then AB is in $\mathrm{SL}_2(\mathbb{R})$.
- Associativity: For any $A, B, C, \in \mathrm{SL}_2(\mathbb{R})$, $(AB)C = A(BC)$.
- Identity: The identity matrix I_2 is in $\mathrm{SL}_2(\mathbb{R})$.
- Inverse: Existence of inverses is guaranteed since $\det(A) \neq 0$ and $\det(A^{-1}) = 1$.

Lie algebra

Theorem. *For any* $X \in M_n(\mathbb{R})$*, then* $\det(\exp(X)) = \exp(\mathrm{Tr}\, X)$.

The space of trace-free matrices in $M_2(\mathbb{R})$ is spanned by

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Local coordinates

We use the above result to endow $\mathrm{SL}_2(\mathbb{R})$ with local coordinates. We will consider the parametrization $\Psi(\theta, x, y)$ of $\mathrm{SL}_2(\mathbb{R})$ given by

$$g := \Psi(\theta, x, y) = \exp(\theta E_2)\exp(x E_1)\exp(y E_3).$$

The vector space spanned by $\{E_1, E_2, E_3\}$ are called the **Lie algebra** of $\mathrm{SL}_2(\mathbb{R})$ denoted by $\mathfrak{sl}_2(\mathbb{R})$ [4]. Commutators are introduced on Lie algebra associated with matrix groups:

$$[A, B] = AB - BA.$$

for any $Y_1.Y_2 \in \mathfrak{sl}_2(\mathbb{R})$, with the commutation relations

$$[E_1, E_2] = -2E_3, \; [E_1, E_3] = -2E_2, \; [E_2, E_3] = -2E_1.$$

Determine the Geodesic Flow

The cotangent bundle $T^*\mathrm{SL}_2(\mathbb{R})$ is an auxilary space that we use to define the geodesic flow. We parametrize $T^*\mathrm{SL}_2(\mathbb{R})$ by the coordinates: $(p, g) := (p_\theta, p_x, p_y, \Psi(\theta, x, y))$.

Hamiltonian Formalism

A function H on $T^*\mathrm{SL}_2(\mathbb{R})$ defines a system of differential equations called Hamilton equation:

$$\begin{pmatrix} \dot p_\theta \\ \dot p_x \\ \dot p_y \end{pmatrix} = -\begin{pmatrix} \frac{\partial H}{\partial \theta} \\ \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial y} \end{pmatrix} \qquad \begin{pmatrix} \dot\theta \\ \dot x \\ \dot y \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial p_\theta} \\ \frac{\partial H}{\partial p_x} \\ \frac{\partial H}{\partial p_y} \end{pmatrix}$$

A solution to the Hamiltonian equations parametrizes the level of the Hamiltonian function.

Poisson Bracket

The Poisson bracket $\{\cdot,\cdot\} : C^\infty(T^*\mathrm{SL}_2(\mathbb{R})) \times C^\infty(T^*\mathrm{SL}_2(\mathbb{R})) \to C^\infty(T^*\mathrm{SL}_2(\mathbb{R}))$ is given by

$$\{f_1, f_2\} = \frac{\partial f_1}{\partial \theta}\frac{\partial f_2}{\partial p_\theta} - \frac{\partial f_1}{\partial p_\theta}\frac{\partial f_2}{\partial \theta} + \frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial p_x} - \frac{\partial f_1}{\partial p_x}\frac{\partial f_2}{\partial x} + \frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial p_y} - \frac{\partial f_1}{\partial p_y}\frac{\partial f_2}{\partial y}.$$

An alternative way of writing Hamilton equations is $\{f, H\} = \dot f$ [1].

Momentum Functions

We define a change of coordinates from $p = (p_\theta, p_x, p_y)$ to $P = (P_1, P_2, P_3)$ given by

$$\begin{aligned} P_1 &= -\frac{\sinh(2y)}{\cosh(2x)}p_\theta + \cosh(2y)p_x - \sinh(2y)\tan(2x)p_y; \\ P_2 &= \frac{\cosh(2y)}{\cosh(2x)}p_\theta - \sinh(2y)p_x + \cosh(2y)\tan(2x)p_y; \\ P_3 &= p_y. \end{aligned}$$

The coordinates (P_1, P_2, P_3) are called momentum functions, they are auxilary functions to build the geodesic flow [9].

The momentum functions satisfies the following Poisson bracket relation

$$\{P_1, P_2\} = -2P_3, \;\; \{P_2, P_3\} = -2P_1 \;\text{ and }\; \{P_1, P_3\} = -2P_2.$$

Geodesic Equations

The Hamiltonian function governing the geodesic flow is given by

$$H(P, g) = \frac{1}{2}\left(P_1^2 + P_2^2 + P_3^2\right).$$

The geodesic equations are the Hamiltonian equations defined by the above equation. If $(p(t), g(t))$ is a solution to the geodesic equations, then $g(t)$ is a geodesic curve in $\mathrm{SL}_2(\mathbb{R})$. When we set up the energy $H = \frac{1}{2}$, then geodesic $g(t)$ is parametrized by arc length.

We write the geodesic equations in terms of the mometum functions since they encode the symmetries of the system

$$\begin{pmatrix} \dot P_1 \\ \dot P_2 \\ \dot P_3 \end{pmatrix} = \begin{pmatrix} 4P_2P_3 \\ 0 \\ -4P_1P_2 \end{pmatrix}, \quad \begin{pmatrix} \dot\theta \\ \dot x \\ \dot y \end{pmatrix} = \begin{pmatrix} 0 & \frac{\cosh(2y)}{\cosh(2x)} & -\frac{\sinh(2y)}{\cosh(2x)} \\ 0 & -\sinh(2y) & \cosh(2y) \\ 1 & \cosh(2y)\tanh(2x) & -\sinh(2y)\tanh(2x) \end{pmatrix}\begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}.$$

Geodesic Flow

The geodesic flow is a map $\Phi^t : T^*\mathrm{SL}_2(\mathbb{R}) \to T^*\mathrm{SL}_2(\mathbb{R})$ defined in the following way: $\Phi^t(P, g) = (P(t), g(t))$, where $(P(t), g(t))$ is a solution to the geodesic equation with initial condition $(P(0), g(0)) = (P, g)$ [5].

The surface $\mathbb{S}(t, g_0)$ which flows along the geodesics, is given by:

$$\mathbb{S}(t; g_0) := \left\{ g \in \mathrm{SL}_2(\mathbb{R}) : (P, g) = \Phi^t(P(0), g_0) \;\text{ and }\; P_1^2(0) + P_2^2(0) + P_3^2(0) = 1 \right\}.$$

Visualizing the Time Evolution

Using numerical ODE-system solver *scipy.integrate.solve ivp*, we are able to numerically solve this system for $t \in [0, 2]$, which gives us how the geodesic curve in coordinate representation in $\mathbb{R}^3$ changes with the parametrization t.
The initial conditions of this numerical solution we present are as follows:

$$\begin{pmatrix} \theta(0) \\ x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} P_1(0) \\ P_2(0) \\ P_3(0) \end{pmatrix} = \begin{pmatrix} \cos(\psi)\cos(\phi) \\ \sin(\psi)\cos(\phi) \\ \sin(\phi) \end{pmatrix}.$$

where $\phi \in (-\pi/2, \pi/2)$, $\psi \in (0, 2\pi)$. In other words, the initial conditions of P_1, P_2, P_3 are parametrized as points on $\mathbb{S}^2$.

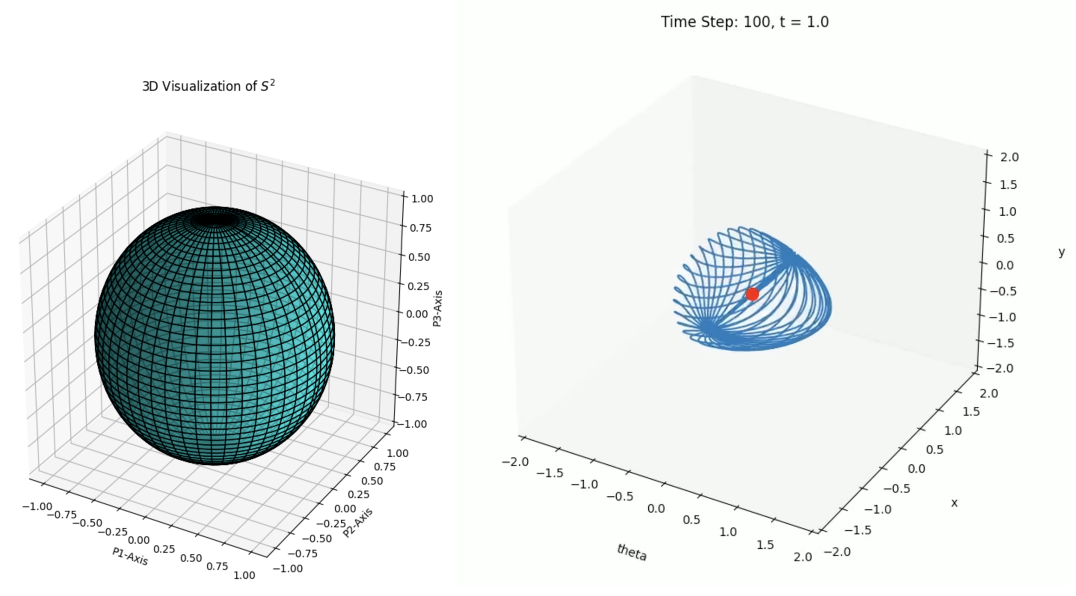

Figure 1: *(left): A visualization of the Euclidean sphere in* $\mathbb{R}^3$*. Each 3-tuple of* $(P_1(0), P_2(0), P_3(0))$ *can be seen as a point on this sphere; (right): A snapshot of the geodesic flow at* $t = 1.0$.

We generate a grid of points in spherical coordinates, with azimuthal angle ϕ ranging from 0 to 2π and polar angle θ ranging from 0 to π.
The resulting points are plotted as a surface in 3D to visualize the unit sphere S^2.
A full video of the geodesic flow is readily presented following the QR code below:

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