Automorphisms of the Fine Curve Graph on Planar Surfaces

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Goal

Find a combinatorial model for the homeomorphism groups of surfaces.

Introduction

Let S be a surface. The **fine curve graph** $C^\dagger(S)$ is the graph with essential, simple, closed curves as vertices and pairs of disjoint curves as edges. Long-Maraglit-Pham-Verberne-Yao [3] show that the automorphism group of the fine curve graph of a closed surface is isomorphic to the homeomorphism group of that surface. We seek to extend this to orientable surfaces with boundary.

Key Definitions

- **Curve** ([4]) - the image of an embedding: $S^1 \to S$.
- **Essential** ([4]) - a curve is essential if it is not homotopic to a point or boundary component
- **Fine Curve Graph** ($\mathcal{C}^\dagger(S)$) ([1])
  - Vertices - essential curves
  - Edges: pairs of disjoint curves
- **Extended Fine Curve Graph** ($\mathcal{EC}^\dagger(S)$) ([3])
  - Vertices - curves
  - Edges: pairs of disjoint curves

Main Conjecture

Let S be a connected planar surface with sufficiently many boundary components. Then the natural map:
η: Homeo(S) -> Aut($C^\dagger(S)$)
is an isomorphism.

Illustration of Fine Curve Graphs

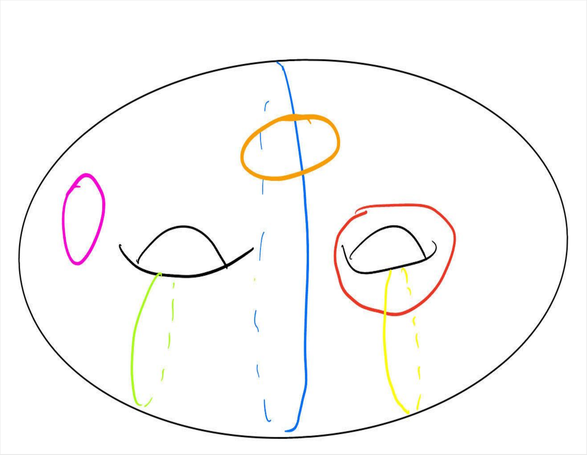
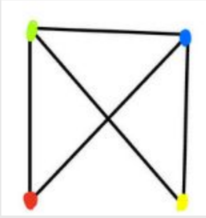
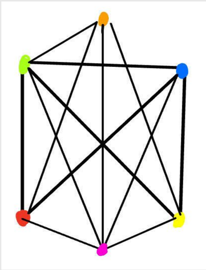


Figure 1: Example of Surface



A part of the fine curve graph



A part of the extended fine curve graph

Proof Outline

Homeo(S)

Proved using convergent sequences of curves

Automorphism of $\mathcal{EC}^\dagger(S)$

Homeomorphisms send curves to curves and preserve disjointness of curves

Shown using preservation of bigon pairs (in progress)

Automorphism of $\mathcal{C}^\dagger(S)$

An automorphism of $\mathcal{C}^\dagger(S)$ extends to an automorphism of $\mathcal{EC}^\dagger(S)$ if it preserves relationships between inessential curves

$\Downarrow$

We want to find a way to characterize inessential curves on our surface using essential curves.

ANSWER: Bigon Pairs

Lemma 1

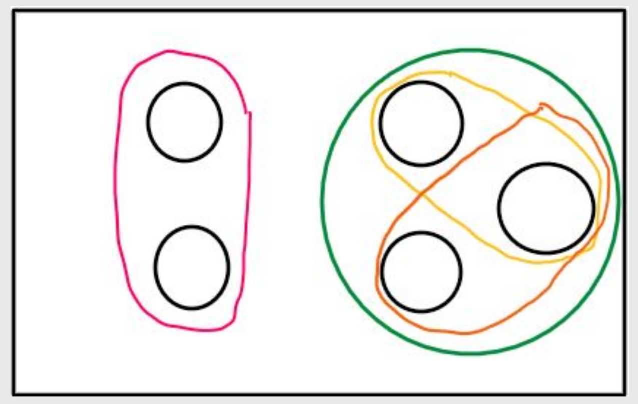
Let c,d be a pair of curves in $\mathcal{C}^\dagger(S)$. Automorphisms of the fine curve graph preserve hulls.
A curve e in $\mathcal{C}^\dagger(S)$ lies in the hull of c,d if and only if the link of e contains the link of c,d.

More Definitions

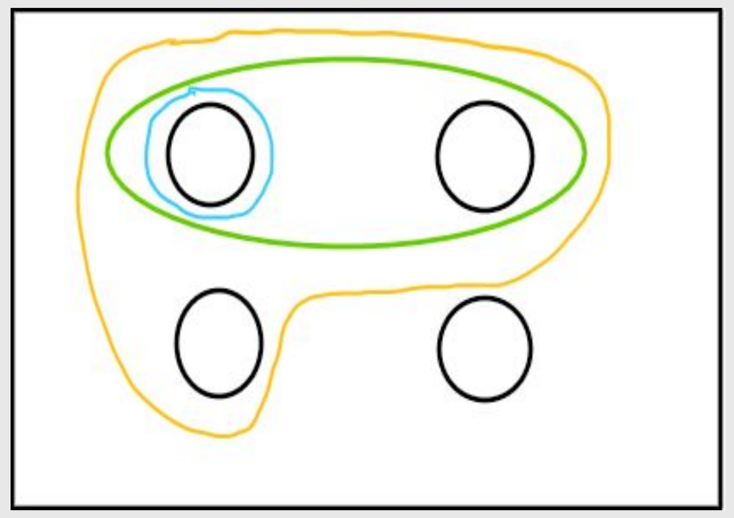
- **Link of a pair of curves c,d** ([2]) - Induced subgraph of vertices adjacent to both c and d.
- **Join** ([2]) - A graph is a join if its vertices can be partitioned into two sets A and B such that every vertex in A shares an edge with every vertex in B.
- **Separating curve c** ([4])- A curve c is separating if S\c is not connected.
- **Hull of a pair of curves c,d** ([3]) - A set including the unions of the curves along with any curves inside disks or annuli bounded by c,d.
- **Convergent Sequence of curves** ([3]) - A sequence of curves (c_i) in $\mathcal{EC}^\dagger(S)$ is convergent to a point if any neighborhood of that point contains all but finitely many elements in the sequence.

Lemma 2

Let A be a separating collection of curves in $\mathcal{C}^\dagger(S)$. Automorphisms of the fine curve graph preserve sides, i.e. for any two curves c,d on one side of A, α(c), α(d) are also on the same side.

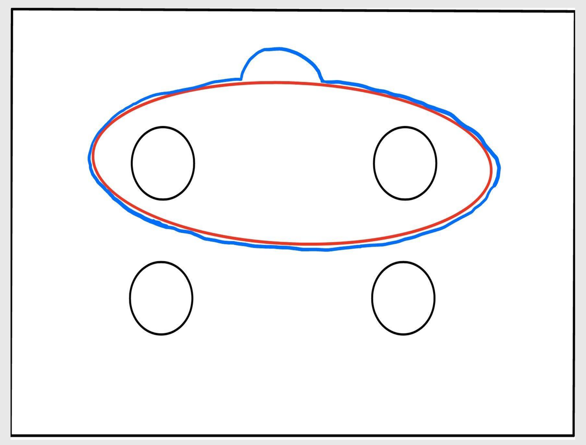


Proof of Lemma 2

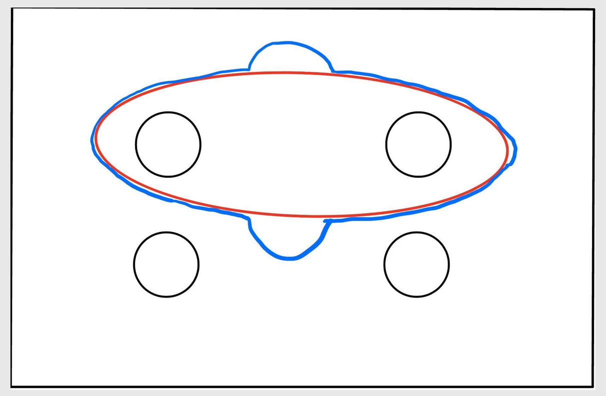


Nested sequence of Curves

By using nested sequences of curves, we can deduce how many boundary components a curve surrounds by what side of the sequence the curve is on.



Bigon Pair



Not a Bigon Pair

We have to ensure we can distinguish bigon pairs in our fine curve graph

Observation: Curves c,d in $C^\dagger(S)$ are a bigon pair if:
- They bound only 2 boundary components;
- The hull of c, d is unique and contains other curves in $C^\dagger(S)$.

Convergent Sequence of Curves

$$\begin{tikzcd} \text{curves } \{c_i\} \arrow[r, "\alpha"] \arrow[d, "\lim"] & \text{curves } \{\alpha(c_i)\} \arrow[d, "\lim"] \\ \text{point } c \arrow[r, "f_\alpha"] & \text{point } f_\alpha(c) \end{tikzcd}$$

Encoding points using sequences of Curves

Next Steps

We aim to complete the proof of our main conjecture. This involves the following.

- Verify that if we use two different pairs to characterize the same bigon, they will continue to characterize the same bigon after applying an automorphism of $\mathcal{C}^\dagger(S)$.
- Find the smallest number of boundary components for which this theorem is true.

Moreover, we hope to generalize the result to arbitrary orientable surfaces with boundary.

References

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