Statistical Analysis of Rank & Bias

Ethan Harr, Gage Larson, Yihan Yao
Mentor: Amichai Lampert

Background

Finite Field F_p:

- Mathematical Field with p elements
- i.e. Only 0, 1, ..., $p-1$ can be used in coefficients & variables of polynomials
- Achieve it by using "mod p".

Example:

$$P = 4x^2yz + z^4 + u^2xy + 3x^2y^2$$

DEFINITION For Rank:

1. Factorize:
   $P = z(4x^2y + z^3) + xy(u^2 + 3xy)$
   *Least number of factorizable components
2. Count:
   2 pieces $\rightarrow$ **A + B** $\rightarrow$ rank = 2

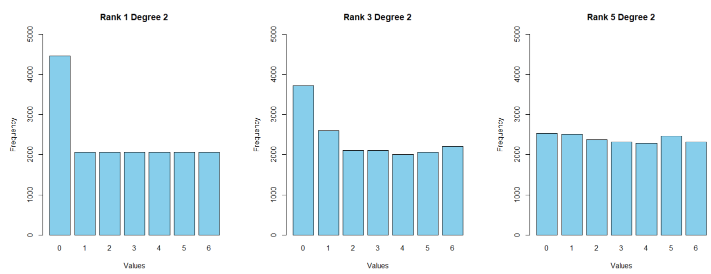
DEFINITION For Bias:

1. Possible values for [x, y, z, u]: $[0,0,1,2], [0,1,2,3], [2,3,1,4], \ldots$
2. Plug in -> Average.
3. Bias: quantify how uniform the distribution of outputs is

Two Properties of Polynomials: Are they related? or are famous mathematicians wrong?

Degree 2; Previously Proven Case

From [Green, Tao] [1], in the degree 2 case: "lack of equidistribution implies bounded rank"



Rank 1 Rank 3 Rank 5 [a]

higher rank $\Longrightarrow$ equidistribution

Degree 3 and up

???

Conjecture: similar relationship for arbitrarily large degree[3]. But what do we see when we actually calculate these values for high degree polynomials?

[a] The polynomials used here are $(x+y+z)(u+v)$, $(x^2+y^2+z^2)$, and $(x^2+xy+xz+xu+xv+y^2+yz+yu+yv+z^2+zu+zv+u^2+uv+v^2)$

SCAN ME

← Link to data and code

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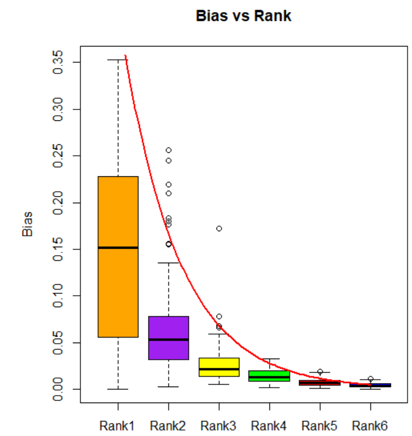
Experimentally!



In degree 3: conjecture seems likely!

Future Direction

$$bias(P) \leq |F_p|^{-r/4} \leq |F_p|^{-r/12}$$

1. Proven (deg 3): $bias(P) \leq |F_p|^{-r/12}$ [2]
   Our examples (deg 3): $bias(P) \leq |F_p|^{-r/4}$
2. Future: fully describe the rank and bias for degree 3 over small fields and a small number of variables
3. This data may help affirm a proper upper bound

References

[1] Ben Green, Terence Tao. *The distribution of polynomials over finite fields, with applications to the Gowers norms*. 2007.

[2] Karim Adiprasito et al. *On the Schmidt and analytic ranks for trilinear forms*. 2021.

[3] Tali Kaufman, Shachar Lovett. *Worst Case to Average Case Reductions for Polynomials*. 2008.

[4] Illinois Geometry Lab. IGL Poster Template. *University of Illinois at Urbana-Champaign Department of Mathematics*, 2017.