Explorations in Gauss and Kloosterman Sums

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What is an Exponential Sum?

Exponential sums are ubiquitous in mathematics. They incorporate a number of diverse mathematical fields and have applications from quadratic reciprocity to heat equations.

A Surprising Example

$$e^{\frac{4i\pi}{5}} - e^{\frac{8i\pi}{5}} - e^{\frac{12i\pi}{5}} + e^{\frac{16i\pi}{5}} = \sqrt{5}$$

This is an example of an exponential sum!

Basic Form

Generally, exponential sums are sums of the following form:

$$\sum_{x\in\mathbb{F}_q^\times} f(x)e^{\frac{2\pi xi}{p}} \quad \text{with} \quad f:\mathbb{F}_q^\times \to \mathbb{C}$$

Project Goal

Explore properties of a variety of exponential sums and provide numerical evidence for their properties. These include questions such as:

- How do certain sums distribute?
- How do these sums reflect the behavior of analagous functions?
- How can we visualize the properties of these sums?

Gauss Sums

Many are familiar with the Gaussian Integral: $\int_{-\infty}^{\infty} ae^{\frac{-(x-b)^2}{2c^2}} = ac\sqrt{2\pi}$
We consider an analagous sum over a finite field:

Definition

The *Gauss sum* $G_q:\mathbb{F}_q^\times \to \mathbb{C}$ is defined as:

$$G(\chi,\psi) = \sum_{x\in\mathbb{F}_q^\times} e^{\frac{2\pi ibj}{q-1}} \cdot e^{\frac{2\pi iax}{p}}$$

In particular, j satisfies $x=\zeta^j$ for some generator ζ.

First, the sizes of Gauss sums are described as follows:

Theorem 1: (Lidl and Niederreiter [3])

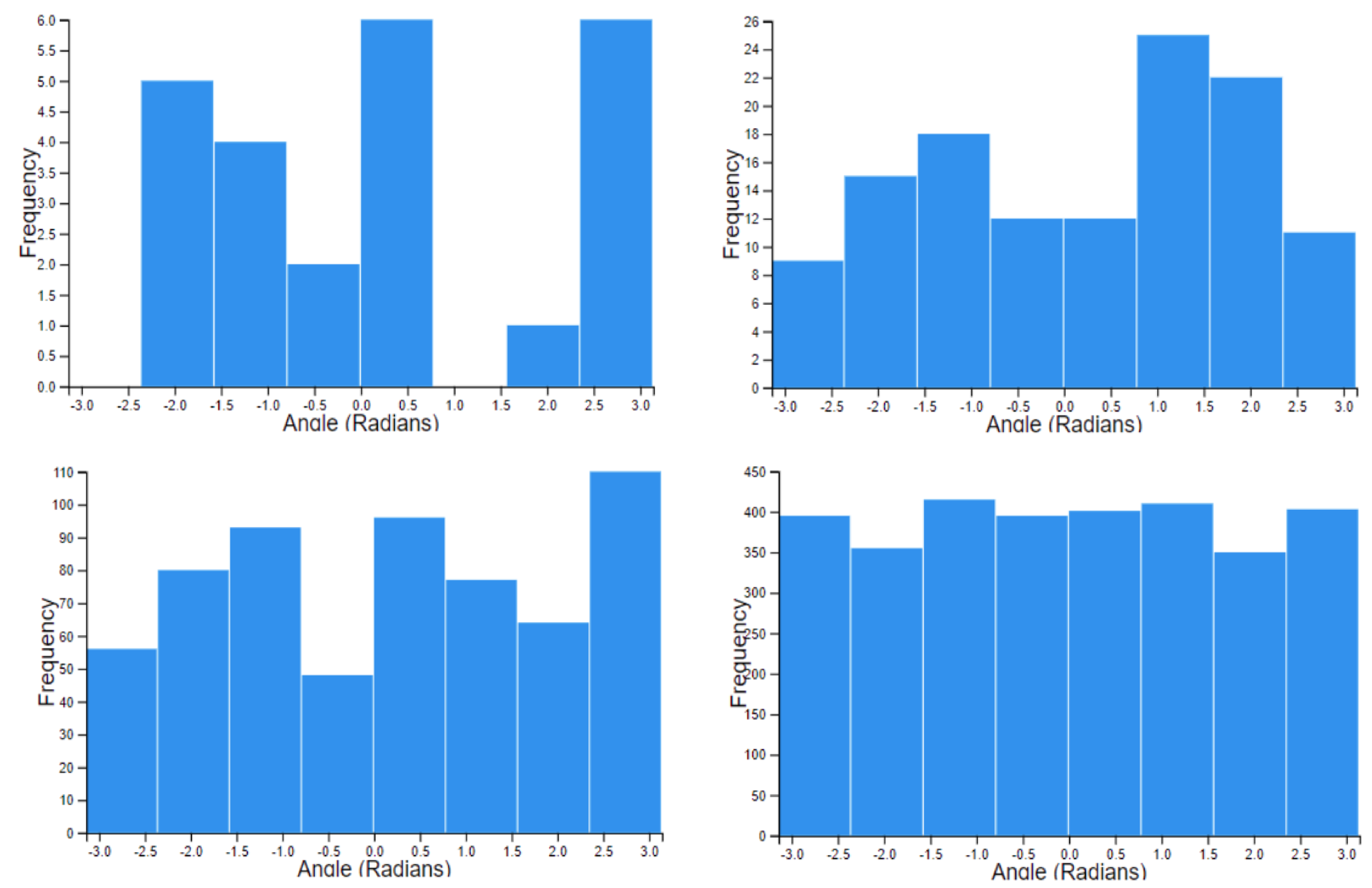
$$G_p(\chi,\psi) = \begin{cases} p-1, & \text{if } \chi=\psi=1 \\ -1, & \text{if } \chi=1, \psi\neq 1 \\ 0, & \text{if } \chi\neq 1, \psi=1 \end{cases}$$

$|G(\chi,\psi)| = \sqrt{p}$ otherwise.

In order to visualize this theorem and calculate the Gauss Sums for user-provided values of the prime characteristic p, field degree d, and a and b, please see the Gauss Sum Calculator HTML code.

Another interesting property of Gauss sums is that their angles tend to equidistribute evenly [2] about the unit circle. See the equidistribution properties in action:

Example 1. *Observe equidistribution among G_{5^d} as the degree grows from 2 to 5:*



To create your own histograms for a specified prime characteristic p, degree d, and value a, please see the Gauss Sum Histogram HTML code.

To test the additional visualization tools with user-provided values, please see the Hasse-Davenport Relation visualizer HTML code. For more interesting applications of Gauss sums, see [1].

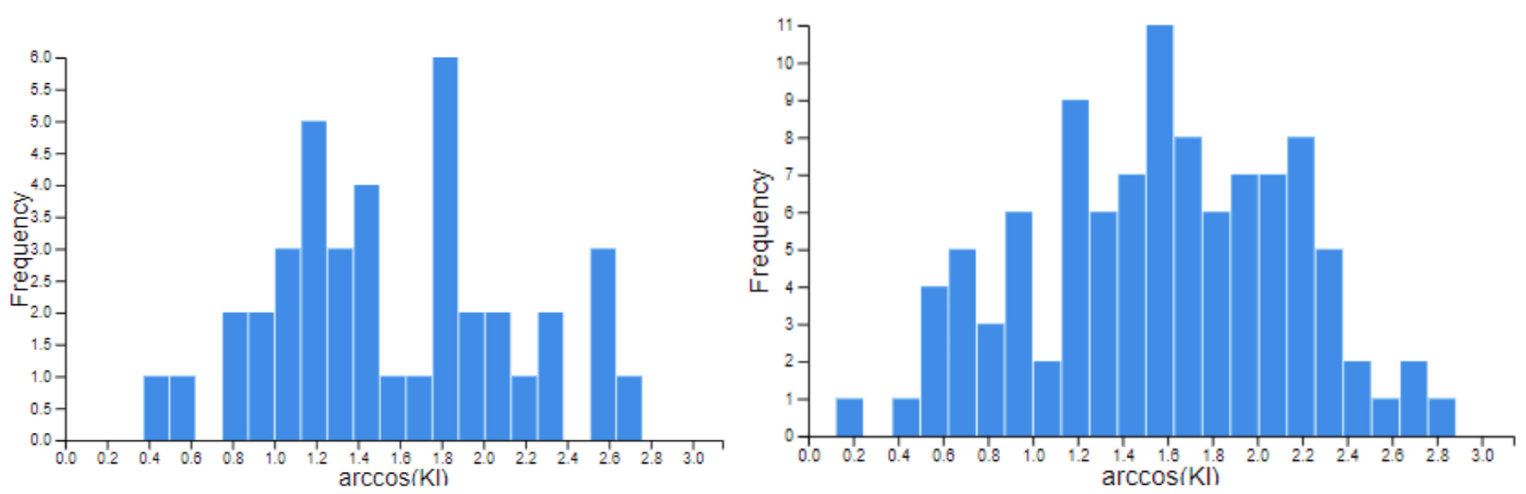
Kloosterman Sums

We can now generalize Gauss sums to a broader class of exponential sums known as Kloosterman sums. One sum of interest is as follows:

Example 2. *The Kloosterman sum* $\frac{1}{\sqrt{q}}\sum_{x\in\mathbb{F}_q^\times}\psi(x+a/x)$ *is analogous to the classical Bessel function* $\int_{\mathbb{S}^1}\exp(x+a/x)\frac{dx}{x}$.

One interesting result [4] is that the distribution of the inverse cosine of these sums resembles $\frac{2}{\pi}\sin^2 x$. See this equidistribution in action:

Example 3. *The above Kloosterman sum over $\mathbb{F}_{41}^\times$ and then $\mathbb{F}_{103}^\times$:*



One tool for further analysis of exponential sums is the L-function, which is as follows.

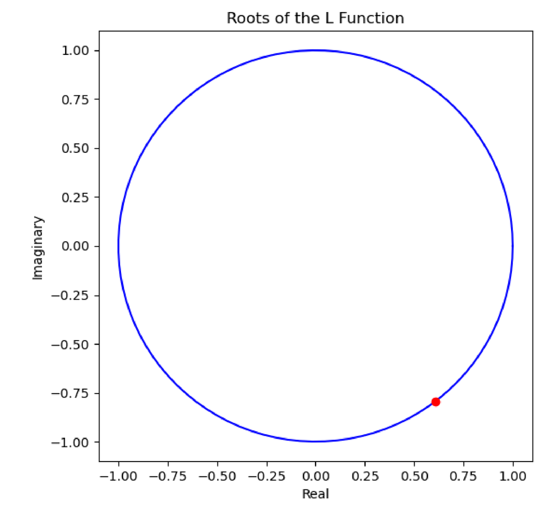
Definition

$$L(T,\widetilde{Kl}) = \exp\Big(\sum_{m\geq 1}\frac{\widetilde{Kl}_m\cdot T^m}{m}\Big)$$

Here, $\widetilde{Kl}_m$ represents the normalized Kloosterman sum over a finite field of degree m.

We can numerically verify that the $L^{(-1)^k}$ is a polynomial of degree q. We can also study the roots of the polynomial:

Example 4. *See the $L-$function root for G_{5^2} and coefficients for Kl_3:*



	Coefficient Value
r = 0	1.0000000000000000000000000000
r = 1	-0.500000000000000 - 0.288675134594813*I
r = 2	0.500000000000000 - 0.288675134594813*I
r = 3	-1.00000000000000 + 1.38777878078145e-17*I
r = 4	-1.69135538907739e-17 + 1.90819582357449e-17*I
r = 5	2.01119502996061e-17 - 1.33573707650214e-16*I

As seen above, the root for G_5^2 lies on the unit circle and the coefficients of degree ≥ 3 are zero for Kl_3.

Our Code

Check out our code repository at `https://gitlab.eecs.umich.edu/logm/wi24/expsums`! We plan to continue to develop more features and deploy more optimizations in coming months.

References

[1] Ellen Eischen and Stephan Garcia. "A Gallery of Gaussian Periods". In: *Proceedings of Bridges 2020: Mathematics, Art, Music, Architecture, Education, Culture.* Ed. by Carolyn Yackel et al. Phoenix, Arizona: Tessellations Publishing, 2020, pp. 243–248. ISBN: 978-1-938664-36-6. URL: `http://archive.bridgesmathart.org/2020/bridges2020-243.html`.

[2] Nicholas M. Katz. *Gauss Sums, Kloosterman Sums, and Monodromy Groups. (AM-116).* Princeton University Press, 1988. ISBN: 9780691084336. URL: `http://www.jstor.org/stable/j.ctt1b7x80x` (visited on 04/05/2024).

[3] Rudolf Lidl and Harald Niederreiter. "Exponential Sums". In: *Finite Fields.* Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1996, 186â267.

[4] I. E. Shparlinski. *On the distribution of Kloosterman sums.* 2006. arXiv: `math/0608595 [math.NT]`.

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