TITOgraphy: When Order Takes Shape

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ABSTRACT

- Our project develops an algorithmic method to compare TITOs by extracting their structural features and information, and determining their relative position in the weak order.
- The implementation produces consistent comparisons and enables visualization of local Hasse diagrams, i.e, Hasse diagrams of finite induced subposets.

Definition (Partial Order). *A partially ordered set is set P together with a relation $\preceq$ on P that satisfies the following axioms:*

1. $a \preceq a,\ \forall a \in P$ *(reflexivity).*

2. For all $a, b \in P$*, if* $a \preceq b$ *and* $b \preceq a$*, then* $a = b$ *(antisymmetry).*

3. For all $a, b, c \in P$*, if* $a \preceq b$ *and* $b \preceq c$*, then* $a \preceq c$ *(transitivity).*

Definition (Total Order). *Partial order $\preceq$ is a* ***total order*** *if for all* $a, b \in P$*,* $a \preceq b$ *or* $b \preceq a$

Definition (TITO). *Fix* $n \in \mathbb{Z}_{>0}$*. A translation-invariant total order (TITO) is a total order $\preceq$ on $\mathbb{Z}$ such that for all* $a, b \in \mathbb{Z}$*,* $a \preceq b$ *if and only if* $a + n \preceq b + n$*.*

OBJECTIVE:

- Developing an algorithm to compare TITOs provides a concrete and computational way to study and visualize this structure.
- Implementing a comparator that outputs TITOs' relations and generates visualization.

The main goal of our project is to design algorithms to determine the weak order between two TITOs.

THEORETICAL BACKGROUND

Definition (Block). *A block of* $(\preceq)$ *is a subset I of $\mathbb{Z}$ such that the ordering of I by $(\preceq)$ has no minimal or maximal element, and for any* $a, c \in I$*, the interval* $\{b \mid a \preceq b \preceq c\} \subset I$ *is finite.*

Definition (Waxing/Waning). *Block I is* ***waxing*** *if* $x \preceq x + n$ *for all* $x \in I$*. I is* ***waning*** *if* $x + n \preceq x$ *for all* $x \in I$*.*

Definition (Inversion). *An* inversion *of a TITO* $(\prec)$ *is a reflection index* (a, b) *with* $b \prec a$*. Denote the set of inversions of* $(\prec)$ *by* $N(\prec)$*.*

Definition (Weak Order). $(\prec_1) \leq (\prec_2)$ *if and only if* $N(\prec_1) \subseteq N(\prec_2)$*.*

Example: (n=3)
$\ldots 4 \preceq 1 \preceq -2 \preceq -5 \ldots -3 \preceq 2 \preceq 0 \preceq 5 \preceq 3 \preceq 8 \ldots$
Blocks:
1. $[\ldots 4 \preceq 1 \preceq -2 \preceq -5 \ldots] \implies$ window: $[\underline{1}]$
2. $[\ldots 2 \preceq 0 \preceq 5 \preceq 3 \ldots] \implies$ window: $[2, 0]$

Window notation: $[\underline{1}][2, 0]$

COMPARISON ALGORITHM

Main idea: decompose the inversion set of a TITO into disjoint subsets, and find comparable finite representations for each component.

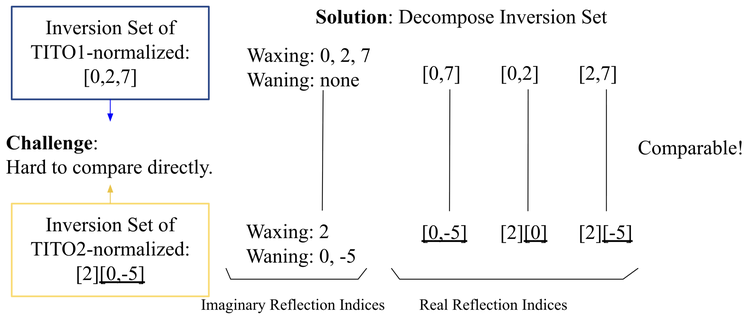


Figure 1: *Main idea behind the comparison algorithm. Here* $n = 3$*.*

NORMALIZATION ALGORITHM: One important pre-processing step in our algorithm is to normalize the window notation so that the following conditions are satisfied:

1. Within each block, the smallest residue class is always at the start.
2. Within each block, the first number is always between 0 and $n - 1$. This provides a unique normalized window notation for each TITO.

Imaginary Reflection Indices: Imaginary reflection indices come from determining, for each residue class, whether it lies in a waxing block or a waning block.
Real Reflection Indices: Real reflection indices come from comparing each pair of residue classes, which can be summarized in the following 6 cases:

Type	Block Normalized Format	Example	Inversion Representation	Inversion Size
Case 1	$[i][j],\ i < j$	$[0][1]$	$\{(-2, 0), (-5, 0), \cdots\}$	Infinite
Case 2	$[i][j],\ i > j$	$[1][0]$	$\{(0, 1), (0, 4), \cdots\}$	Infinite
Case 3	$[i, j],\ i < j$	$[0, 7]$	$\{(0, 1), (0, 4)\}$	Finite
Case 4	$[i, j],\ i > j$	$[0, -2]$	$\{(-2, 0)\}$	Finite
Case 5	$[\underline{i, j}],\ i < j$	$[\underline{0, 4}]$	complement of $\{(0, 1), (0, 4)\}$	Cofinite
Case 6	$[\underline{i, j}],\ i > j$	$[\underline{0, -5}]$	complement of $\{(-2, 0)\}$	Cofinite

Table 1: *Description of inversions generated by two residue classes in each case.*

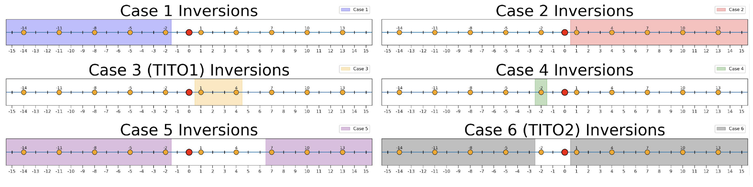
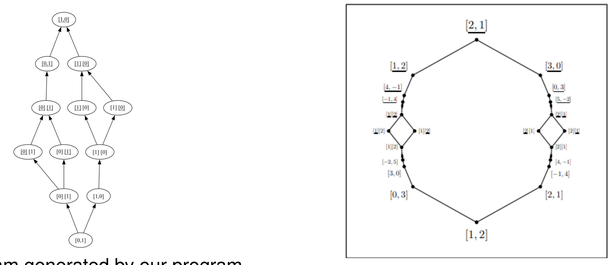


Figure 2: *Inversions generated by two residue classes.*

HASSE DIAGRAM



(a) Diagram generated by our program

(b) Diagram from Barkley[1]

Figure 3: *Hasse diagrams for* $n = 2$ *in two representations.*

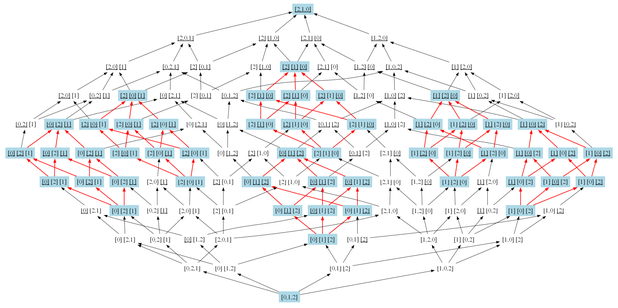


Figure 4: *Hasse Diagram for* $n = 3$ *with elements* $0, 1, 2$*.*

CONCLUSION

We can use a finite representation for the infinite structure of the inversion set of TITO. We have accomplished:

- Implementing a complete TITO comparison algorithm
- Correctly identifying all four possibilities: $=, <, >,$ incomparable
- Verifying consistency with theoretical properties
- Generating Hasse diagrams for small order-ideals of TITOs

FUTURE WORK

- Complete the algorithm to compute the least upper bound (join) of TITOs
- Extend the Hasse diagram to include elements beyond the initial residue classes 0 through $n - 1$.

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