Brownian Game

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LOG(M)

ABSTRACT

- Given a portfolio consisting of risky assets (stocks) and risk-free assets (bonds), how can we make the most optimal investment decisions under conditions of uncertainty using stochastic calculus tools
- Stock prices and wealth are modeled in continuous time with Brownian motion and stochastic differential equations (SDEs)
- Using monte carlo simulations, we explore how risk preferences affect the optimal fraction of wealth invested in each asset
- For simplicity, we start with a simple portfolio consisting of one risky asset and one risk-free asset

OBJECTIVE:
Model financial assets under uncertainty and use stochastic control to determine optimal portfolio allocations across different levels of risk aversion.

THEORETICAL BACKGROUND

How Discrete Price Fluctuations Become Continuous Models
We start with a discrete-time model for prices or wealth with increments

$$X_{t+\Delta t} - X_t = \mu(t, X_t)\,\Delta t + \sigma(t, X_t)\,\Delta Z_t,$$

changes over short intervals have a drift term and a random shock. As the time step becomes smaller, this converges to a continuous-time model driven by Brownian motion.

Definition. ***Wiener process (Brownian motion).*** *A process $(W_t)_{t\geq 0}$ is a Wiener process if $W_0 = 0$, it has almost surely continuous paths, independent increments, and for $0 \leq s < t$, $W_t - W_s \sim \mathcal{N}(0, t-s)$.*

Definition. ***Stochastic differential equation (SDE).*** *An SDE driven by Brownian motion has the form*

$$dX_t = \mu(t, X_t)\,dt + \sigma(t, X_t)\,dW_t,$$

where μ is the drift and σ is the volatility. This forms the foundation of our continuous-time model for asset dynamics and wealth evolution.

These concepts provide the probabilistic background for the portfolio model and Monte Carlo simulations presented in the rest of the poster.

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MODEL AND METHODS

Market model and wealth dynamics.
We work in a continuous–time complete market driven by Brownian motion. The risk–free asset B_t and risky asset S_t follow

$$\frac{dB_t}{B_t} = r_t\,dt, \qquad \frac{dS_t}{S_t} = \mu_t\,dt + \sigma_t\,dW_t,$$

where r_t is the instantaneous risk-free interest rate, μ_t the expected stock return and $\sigma_t > 0$ the volatility.
If an investor with wealth X_t invests a fraction π_t of wealth in the risky asset, then under self–financing

$$dX_t = \big[r_t X_t + \pi_t X t(\mu_t - r_t)\big]dt + \pi_t X t \sigma_t\,dW_t.$$

This SDE is the starting point both for the classical HJB approach and for our Monte Carlo method.

HJB and the Merton problem.
For CRRA preferences the investor maximizes expected utility of intertemporal consumption c_t and/or terminal wealth X_T:

$$\sup_{\{\pi_t, c_t\}} \mathbb{E}\left[\int_0^T e^{-\rho t}\frac{c_t^{1-\gamma}}{1-\gamma}\,dt \;+\; e^{-\rho T}\frac{X_T^{1-\gamma}}{1-\gamma}\right], \qquad \gamma > 0.$$

Let $V(t,x)$ be the value function. The Hamilton–Jacobi–Bellman equation is

$$0 = \sup_{\pi, c}\Big\{V_t + (rx + \pi x(\mu - r) - c)V_x + \tfrac{1}{2}\pi^2\sigma^2x^2V_{xx} + \tfrac{c^{1-\gamma}}{1-\gamma}\Big\}.$$

In the classical Merton model with constant (r, μ, σ) this PDE has a closed form solution. The optimal risky fraction is constant,

$$\pi_t^{\text{Merton}} = \frac{\mu - r}{\gamma\sigma^2},$$

and the optimal consumption rule is proportional to wealth. In more general models with stochastic r_t and market price of risk, explicit formulas are no longer available and we turn to Monte Carlo methods.

Monte Carlo portfolio algorithm (Cvitanić–Goukasian–Zapatero).
In the paper *Monte Carlo computation of optimal portfolios in complete markets* (2003) the authors work with a richer model where

- the short rate r_t follows a CIR–type process,
- the market price of risk λ_t follows a mean–reverting OU process,
- the pricing kernel (state–price density) ξ_t is an exponential martingale built from λ_t.

Using martingale methods, the optimal wealth X_t^* for a CRRA investor can be written as a conditional expectation of future optimal consumption and/or terminal wealth under the risk–neutral measure. The key identity is that the optimal portfolio π_t^* coincides with the diffusion coefficient of the wealth process:

$$dX_t^* = \cdots + \varphi_t\,dW_t \qquad \Rightarrow \qquad \pi_t^* = \frac{\varphi_t}{\sigma_t X_t^*}.$$

CGZ estimate φ_t by a nested Monte Carlo scheme. For a small Δt,

$$\varphi_t \approx \frac{1}{\Delta t}\,\mathbb{E}\big[(X_{t+\Delta t}^* - X_t^*)(W_{t+\Delta t} - W_t) \mid \mathcal{F}_t\big].$$

This gives us an estimate of the optimal risky fraction π_t^* at each time.

FIGURES

γ	T	π_{IC} (ours)	π_{IC} (paper)	Diff IC	π_{TW} (ours)	π_{TW} (paper)	Diff TW
0.5	1	1.1116	1.042	0.0696	1.1300	1.095	0.0350
0.5	5	0.5204	1.221	-0.7006	0.0943	1.236	-1.1417
0.5	10	1.5034	1.272	0.2314	1.7620	1.279	0.4830
0.0	1	0.4652	0.500	-0.0348	0.4803	0.500	-0.0197
0.0	5	0.5296	0.500	0.0296	0.5377	0.500	0.0377
0.0	10	0.4298	0.500	-0.0702	0.5161	0.500	0.0161
-1.0	1	0.3017	0.244	0.0577	0.3227	0.252	0.0707
-1.0	5	0.2251	0.270	-0.0449	0.2940	0.295	-0.0010
-1.0	10	0.1936	0.297	-0.1034	0.6888	0.328	0.3608
-2.0	1	0.2177	0.174	0.0437	0.1743	0.175	-0.0007
-2.0	5	0.4234	0.210	0.2134	0.9193	0.230	0.6893
-2.0	10	0.2543	0.239	0.0153	1.0505	0.270	0.7805

Table 1: *Horizon and risk aversion. Comparison of our Monte Carlo estimates of the optimal risky fraction π^* for IC and TW investors with the original values reported in Table 1 of Cvitanić–Goukasian–Zapatero (2003).*

RESULTS

- Each row in the table corresponds to one pair (γ, T) and reports the optimal fraction π^* invested in the risky asset for intertemporal consumption (IC) and terminal wealth (TW) investors.
- For moderate risk aversion ($\gamma = -1, -2$) and short horizon ($T = 1$), our portfolios are close to the published values: differences are of order 10^{-2} to 10^{-1}, consistent with fast Monte Carlo settings.
- For longer horizons ($T = 5, 10$) and extreme preferences ($\gamma = 0.5, \gamma = -2$), the differences become larger, illustrating the tradeoff between simulation time and accuracy. Increasing the number of paths in the outer and inner loops reduces these discrepancies.
- Qualitatively, we reproduce the main comparative statics of the paper: TW investors typically allocate more to the risky asset than IC investors with the same (γ, T) and for the published values (paper column) the optimal risky fraction π_t^* increases with the horizon T and with lower effective risk aversion.

APPLICATIONS or FUTURE WORK

- **Practical portfolio guidelines.** Calibrate the interest-rate and equity parameters to historical data and use our Monte Carlo engine to generate a surface $(\gamma, T) \mapsto \pi^*(\gamma, T)$. This would provide rule-of-thumb risky fractions for investors with different risk aversion levels and horizons (for example, target-date retirement planning).
- **Richer market models and diversification.** Extend the Cvitanić–Goukassian–Zapatero framework to multiple risky assets and additional state variables (e.g., stochastic volatility), and use our implementation as a baseline to study how optimal IC/TW portfolios allocate wealth across assets under different correlation structures.
- **Constraints and market frictions.** Incorporate realistic constraints such as no short-selling, leverage limits, or proportional transaction costs, and adapt the martingale/-Monte Carlo algorithm to quantify how these frictions change the optimal risky fraction.
- **Numerical methods beyond Monte Carlo.** Compare our Monte Carlo estimates with other numerical approximations of the HJB value function—such as finite-difference PDE solvers or reinforcement-learning-based controllers—to better understand the strengths and limitations of each approach.