



# Numerical Solutions of the Nonlinear Schrodinger Equation

LOG(M)

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## 1. Background

Partial differential equations (PDEs) are a fundamental tool in mathematics and science because they are incredibly useful for modeling and analyzing a wide range of phenomena in fields such as physics, engineering, economics, biology, etc. When faced with a real world problem, it is often helpful to abstract the problem into a mathematical model which we can solve, which can be mathematically interpreted and applied to the same real world problem. Partial differential equations are such mathematical models.

We investigate the Nonlinear Schrodinger Equation (NLS), which has applications in fiber optics, Bose-Einstein condensates, and deep water and rogue waves.



### Nonlinear Schrodinger Problem

**Definition.** The Nonlinear Schrödinger Equation (NLSE) is given by

$$i\varepsilon u_t + \frac{\varepsilon^2}{2} u_{xx} - \kappa |u|^2 u = 0$$

where the subscripts indicate partial derivatives and  $u(x, t)$  is a complex-valued function s.t.  $u : \mathbb{R} \times [0, \infty) \rightarrow \mathbb{C}$

The equation is said to be focusing if  $\kappa < 0$ , and defocusing if  $\kappa > 0$ .

## 2. Problem

Before we define the problem, we can discuss the physical interpretation of the constant  $\varepsilon$  in the NLSE. In physical applications,  $\varepsilon$  in the NLSE is Planck's constant,  $\hbar$ . Planck's constant represents the point where physical behavior is modeled by quantum mechanics rather than classical mechanics. One of our project's goals is to see how the solutions to the NLSE change as this constant may change.

The main goal of our project is to determine the solutions to the focusing Nonlinear Schrödinger Equation numerically, and in particular, to better understand how solutions with interesting initial conditions behave in the semiclassical limit, i.e., when  $\varepsilon \rightarrow 0$  In particular, we are interested in initial conditions that:

- are even functions, decay exponentially, are analytic, have strictly negative second spatial derivatives

Reasons for studying the focusing case include:

- the equation can only have solutions in the semiclassical limit if the initial conditions are analytic

## 3. Method

To study the Nonlinear Schrodinger Equation, we employ the split-step Fourier method. This method computes the solution in a series of small steps by treating the linear and nonlinear term separately. The NLSE can be solved analytically for some initial conditions by something called the Inverse Scattering Transform, first introduced by Gardner et. al. [4]. There exists a wide variety of initial conditions for which the NLSE has been solved analytically, and we intend to improve our model by comparing our numerical results to the analytical solutions [7].

## 4. Results

We show some results for the semiclassical limit of  $u(x, 0) = \text{sech}(x)$  as  $\varepsilon \rightarrow 0$ .

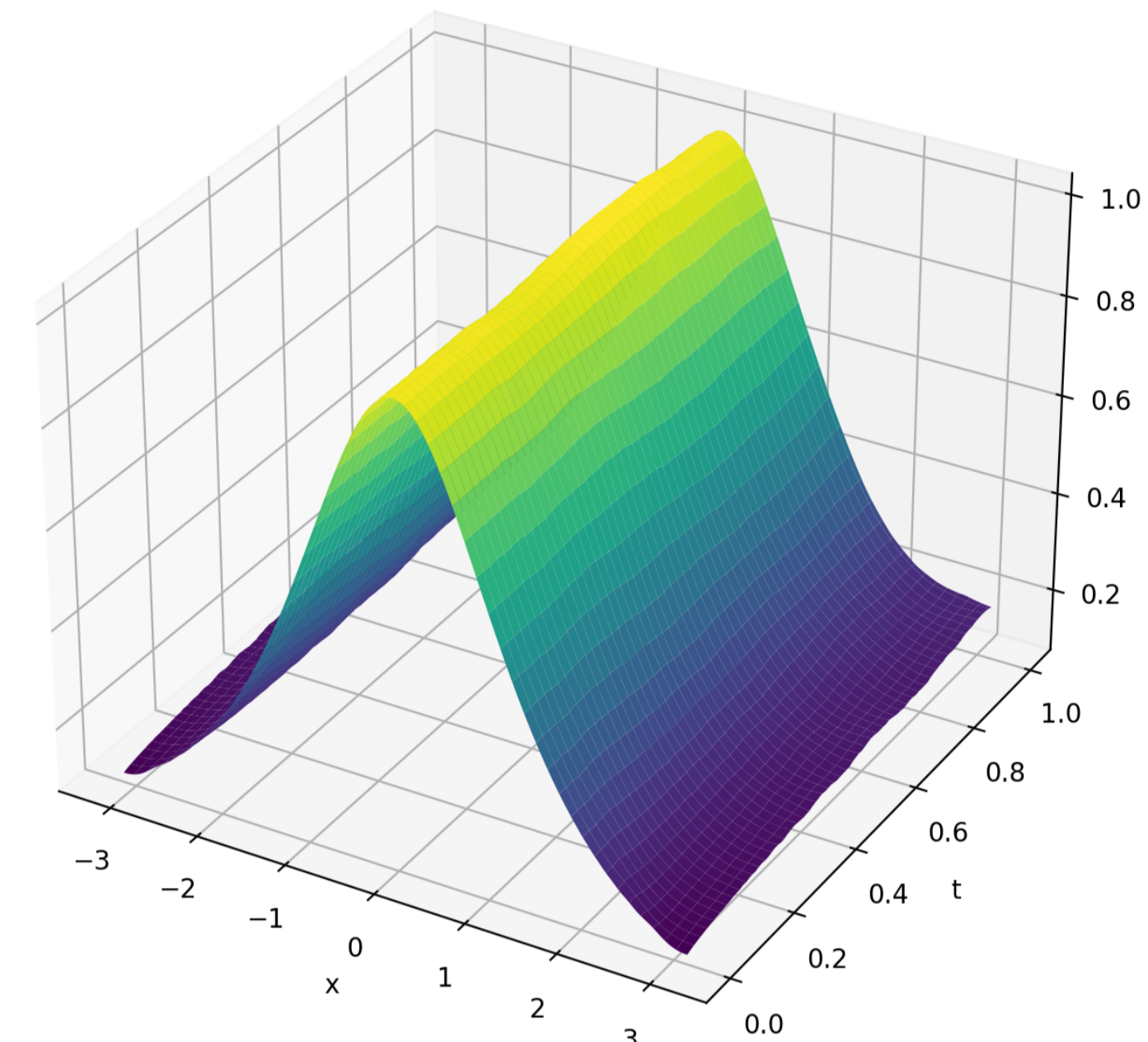


Figure 1:  $u(x, 0) = \text{sech}(x)$ ,  $\varepsilon = 1$

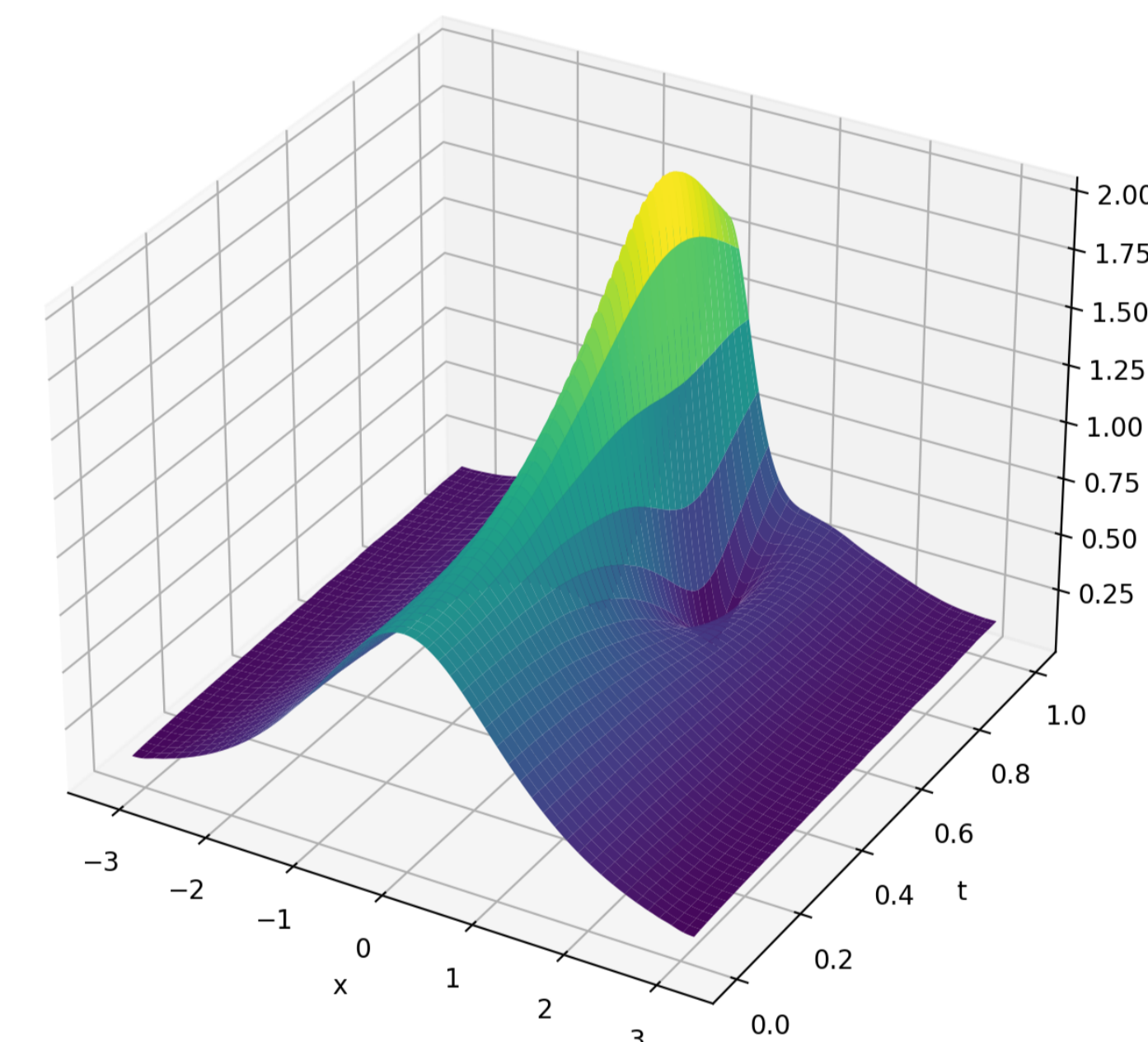


Figure 2:  $u(x, 0) = \text{sech}(x)$ ,  $\varepsilon = 0.5$

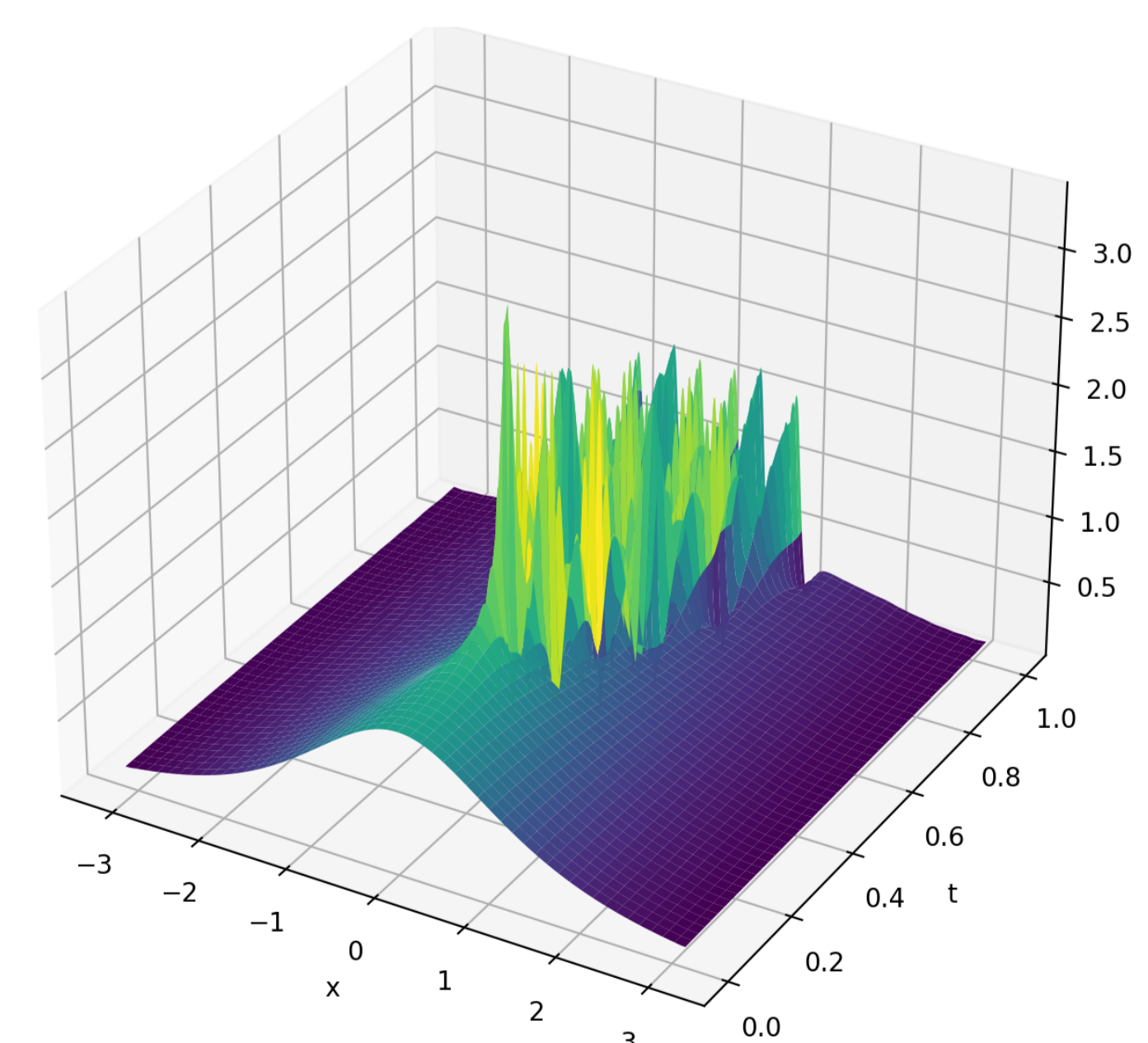
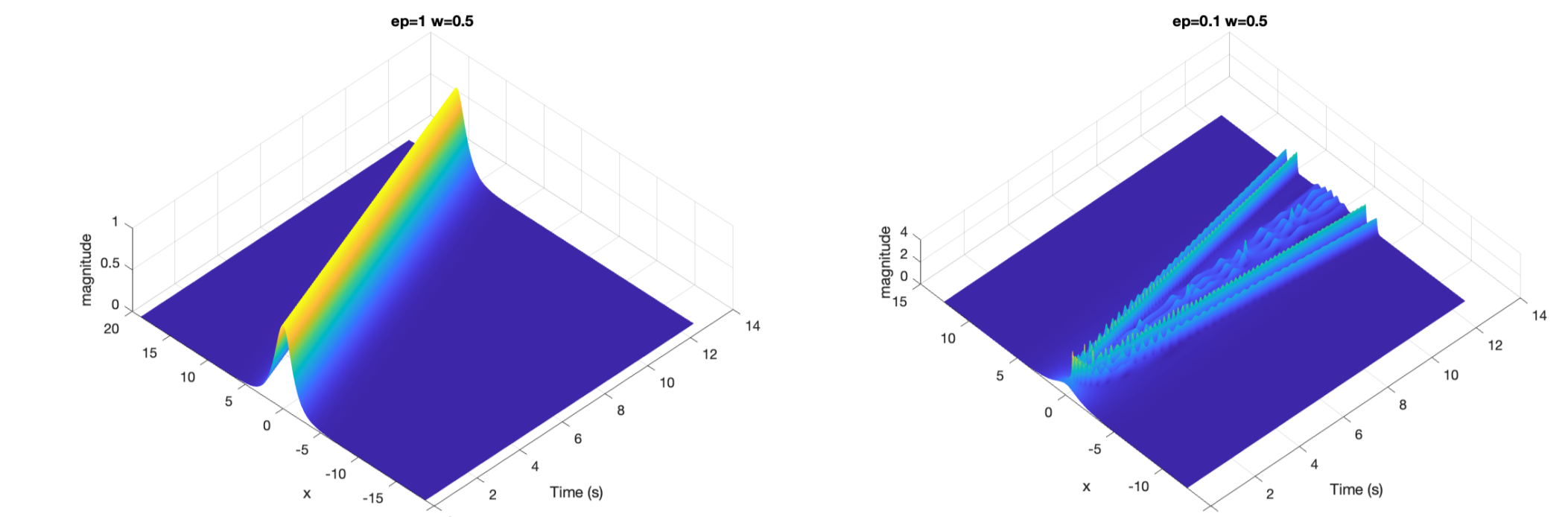


Figure 3:  $u(x, 0) = \text{sech}(x)$ ,  $\varepsilon = 0.1$

We now show some interesting visualizations.



This is sech with a phase velocity. In addition to the mathematical results, this project has also produced code and supplemental documentation to solve the NLSE in both MATLAB and Python. Both the code and documentation will be made public at the conclusion of the project.

## 5. Future Directions

Following the completion of our implementation of a numerical solver of the Nonlinear Schrödinger Equation, we would like to employ the Fourier discrete cosine transform (DCT) to the split-step algorithm, as the initial conditions that are of interest are given by even, analytic functions. Advantages of using DCT compared to FFT include:

- It removes errors due to odd modes
- It is more computationally efficient compared to FFT

We would also like to determine the mean squared errors of the numerical solutions generated by the split-step method using DCT and FFT, against known analytical solutions to the NLSE.

Additionally, we would like to expand our scope to include the modified Nonlinear Schrödinger Equation, which is defined in the following way:

### Modified Nonlinear Schrödinger Equation

$$i\varepsilon u_t + \frac{\varepsilon^2}{2} u_{xx} - \kappa |u|^2 u + \alpha (|u|^2 u)_x = 0$$

A wide variety of modified versions of the Nonlinear Schrödinger Equation exist with applications including

- modelling gravity waves in the presence of wind, dissipation, and shear currents
- the study of how intense light pulses propagate in nanoparticle-doped glass
- modelling Bose-Einstein condensates in ultra-cold temperature

[1], [3], [8]. By studying how to adapt our implementation, we hope to better understand how the NLSE can be changed to model different types of real-world phenomena.

## References

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