Why does the kid swing alone in the park?

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Introduction

Observe a child sitting on a swing. What allows them to move back and forth though they're sitting in the same position? The continued motion of bending and extending their legs! This simple example illustrates the concept of parametric resonance:
Parametric Resonance: a phenomenon that causes an increase in perceived energy in our current system through the translation of external energy from an outside system

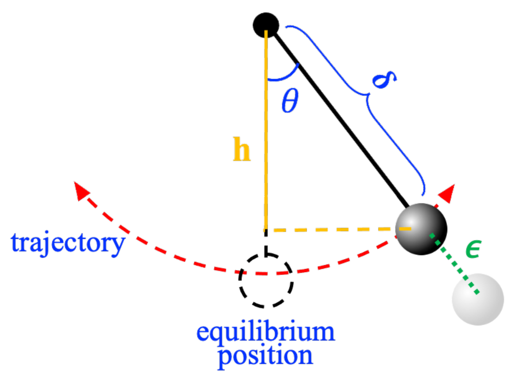
Goal

- Understand the concept of resonance as it relates to a pendulum system through Matheiu's Equation
- Create stability charts for different solutions to Matheiu's Equation using theorems about the trace of a linearity matrix

The concept of resonance relates to that of the kid swinging because it describes how the biological energy of the child translates into mechanical, kinetic energy by moving their body forwards and backwards.

⇐ An illustration of parametric resonance! This video shows monks pulling a rope in a church, causing the increased oscillation of a bell swinging from side to side

Background



Consider a classic pendulum. This is a great example of a nonlinear, autonomous system. How does the angle θ change over time?

We can model the change in this quantity by measuring $\ddot{\theta}$, the acceleration of angle θ over time. Matheiu's equation (depicted below) describes how the pendulum system produces periodic parametric motion.

Mathieu's Equation

Mathieu's equation is a second-order non-autonomous differential equation with the form

$$\ddot{\theta} = -\Big(\delta + \epsilon\omega(t)\Big)\theta(t)$$

We're interested in analyzing the behavior of the equation when $\omega(t) = \cos(2t)$

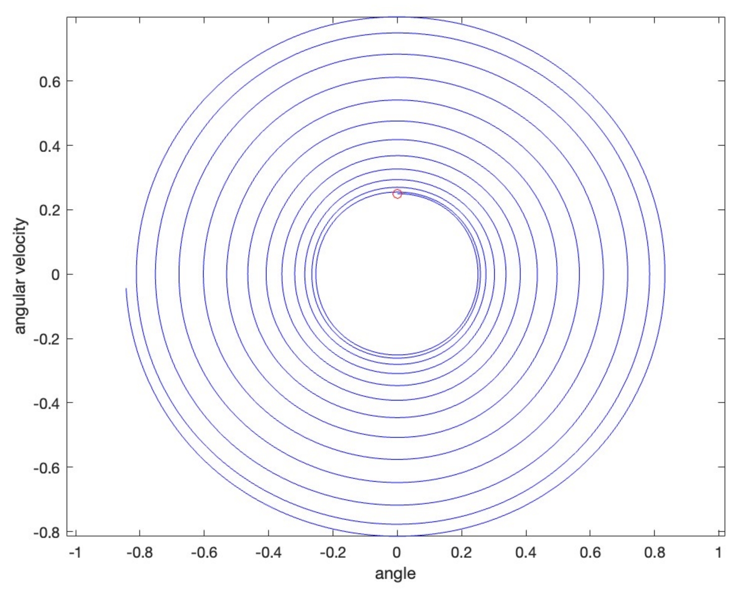
What does Mathieu's Equation represent?

- An ordinary differential equation relating θ, the angle of the pendulum, with time t
- Models the behavior of a kid swinging their legs with frequency $\omega(t)$ (We choose $\omega(t) = \cos(2t)$ because 2 is the constant that allows for the natural periodicity of resonance to exhibit itself in the clearest manner)
- Acceleration is related to both the original length of the pendulum δ and the displacement of the pendulum's length from its original position ϵ

Initial Exploration

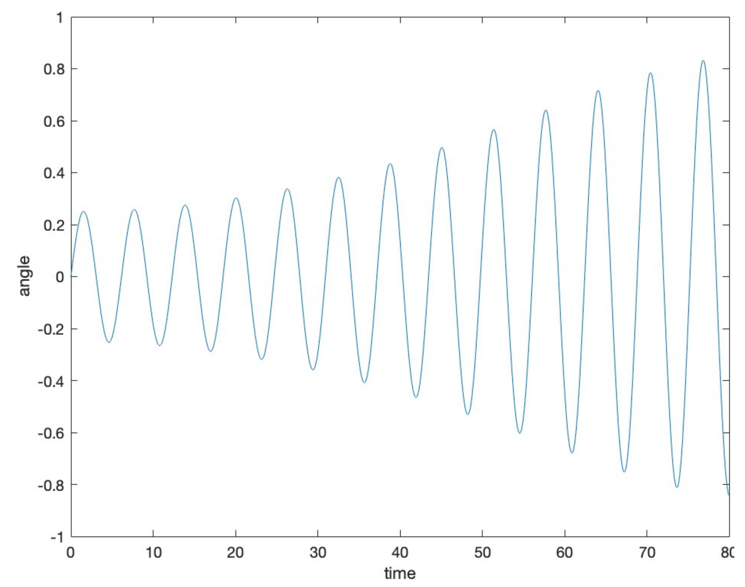
Resonance occurs when energy increases over time. When this occurs, both position and velocity are expected to increase. The plots below confirms this is so:

Figure 1: *Angular Velocity V.S. Angle θ*



The red dot represents theta's initial condition. Observe that both its velocity and actual value are increasing over time.

Figure 2: *Angle θ V.S. Time*



Notice that as energy is imported into the system, the angle becomes greater periodically.

We are now interested in exploring what it means for a particular solution to be stable or unstable for specific values of δ and ϵ.

- **Stable:** when all solutions to the equation are bounded and nonperiodic, i.e. do not diverge to infinity
- **Unstable:** some solutions are unbounded, with chaotic behaviors being exhibited

These definitions become relevant as we explore numerical solutions to different representations of Mathieu's Equation.

Numerical Solutions

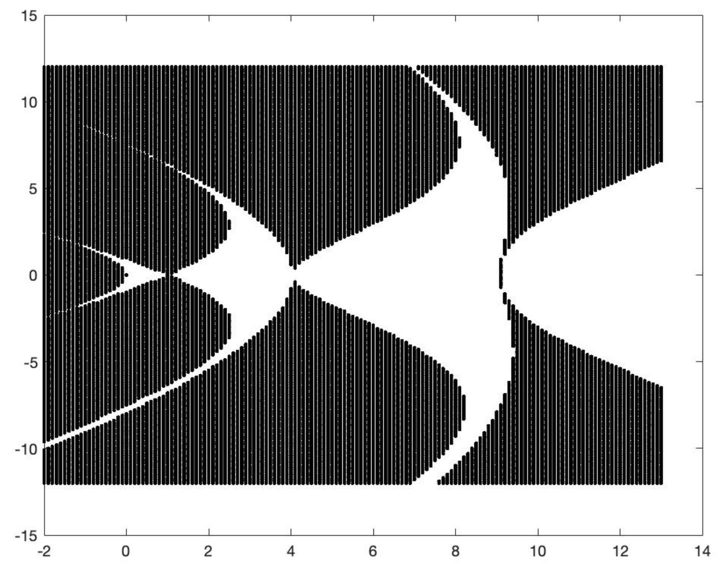
- Recall that Mathieu's equation is a linear map. That means all linear combinations of its solutions will also be solutions. We can then define a linear map from the set of all posible solutions to itself.
- Define the **flow** of Matheiu's equation to be a linear map $A : \mathbf{R}^2 \to \mathbf{R}^2$ that preserves the volume of a solution. Because A is a linear map, it can be modeled by a matrix M. Furthermore, because it preserves volume, $detA = 1$. Thus the solutions to this matrix, which determine the stability of our original equation, can be determined by the **trace** of M.

The following theorem illustrates this finding:

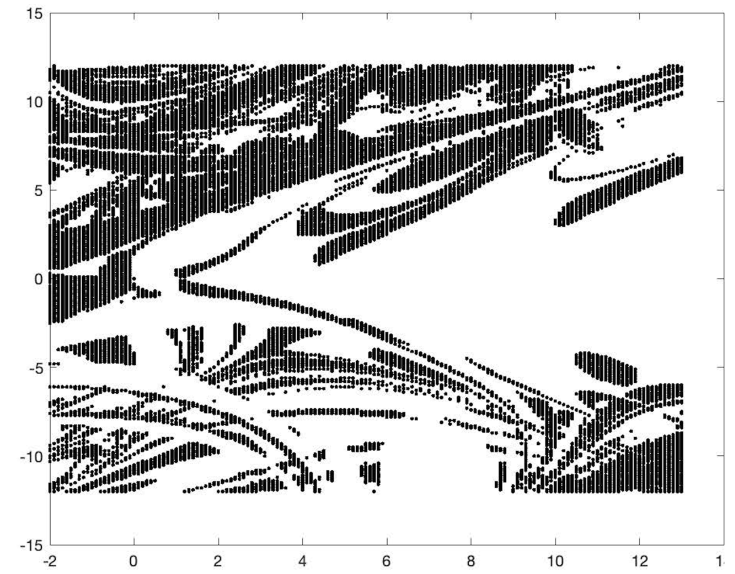
Theorem

Let A be the matrix of a linear mapping of the plane to itself which preserves area ($\det A = 1$). Then the mapping A is stable if $|tr\ A| < 2$, and unstable if $|tr\ A| > 2$.

We developed a computer program to computer matrix M using the numerical solutions to Mathieu's equation for different combinations of ϵ and δ. By computing the trace of matrix M, we were able to determine regions of stability and instability in our graph of ϵ vs. δ. We produced one chart with $\theta(t)$ and one with $sin(\theta(t))$



The stability chart for the reduced Mathieu's equation, where we set $\omega(t) = \cos(\theta)$.



The stability chart for the unreduced Mathieu's equation, where we have $\sin(\theta)$ instead of θ

Future Directions

We can take our project further in the following three ways:

- **Antiresonance**

  Antiresonance refers to the situation where energy is being transported out of the system. We could explore this counterpart to resonance by constructing stability diagrams for antiresonant systems.

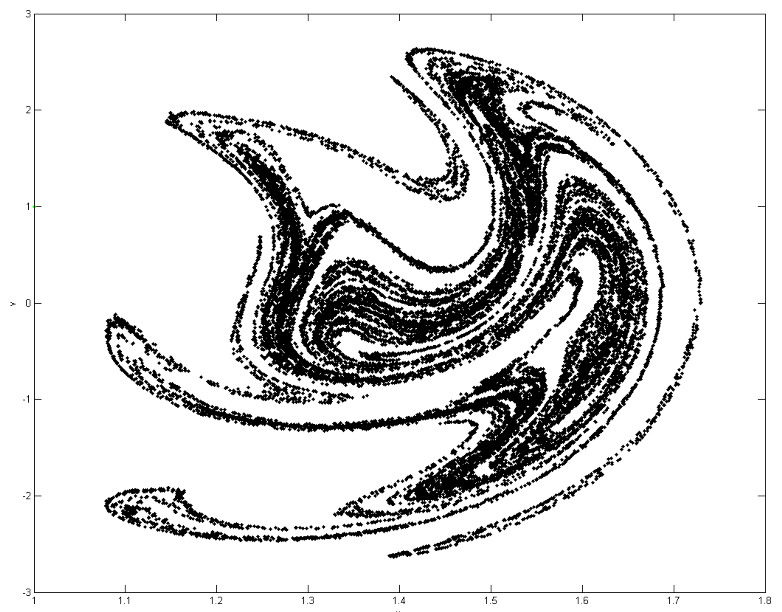
- **Hill's Equation**

  Hill's equation is a second-order linear differential equation that takes the following form:

$$\frac{d^2y}{dx^2} + f(t)y = 0$$

  for some function f that depends on time. It is clear that Mathieu's equation acts on a specific case of Hill's equation; we could potentially explore other forms of $f(t)$ instead of simply restricting to periodic ones like we did in our current investigation.

- **Poincare Maps**



Throughout this project, we've been exploring the values of (ε, δ) to find areas where motion is stable. While we now have a clearer understanding of how the pendulum will behave in the stable case, we have no knowledge of its behavior when it is unstable. We could further our investigation by exploring the unstable case and plotting graphs that showcase motion in such cases. The figure below depicts a Poincare map that plots position vs. velocity in a chaotic system.

References

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