Quantum Pipe Dreams

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Introduction: Schubert Polynomials, Pipes, and Strings

Objective

Construct a combinatorial object to represent Quantum Schubert Polynomials and implement a Python library for pipe-dreams.

Definition. *A permutation $\sigma \in S_n$ is an ordering of numbers $1, 2, \dots, n$. For example, $\sigma = (3,1,4,2) = (\sigma(1), \sigma(2), \sigma(3), \sigma(4))$ is a permutation in S_4 mapping 1 to 3, 2 to 1, 3 to 4, and 4 to 2.*

Schubert Polynomial

- They are indexed by permutation, i.e the Schubert polynomial of σ is $\mathfrak{S}_\sigma$, where σ is some permutation.
- Represent cohomology classes in flag varieties
- More easily manipulated compared to their geometric counterparts.

Quantum Schubert Polynomial

- A generalization of Schubert Polynomials. Extra "quantum" q-variables.
- Have some applications in quantum physics and string theory.

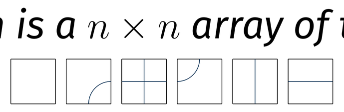
σ	S_σ	S^q_σ
123	1	1
213	x_1	x_1
132	x_1+x_2	x_1+x_2
231	x_1x_2	$x_1x_2+q_1$
312	x_1^2	$x_1^2-q_1$
321	$x_1^2x_2$	$x_1(x_1x_2+q_1)$

Schubert Polynomials can be calculated recursively using symmetric and divided difference operators, and quantum schubert polynomial also have a complex explicit definitions. Combinatorial representations help generate these polynomials without the full algebraic or recursive definitions.

Bumpless Pipe Dreams

Ordinary and bumpless pipedreams are two combinatorial representations of Schubert polynomials, and there is a bijection between these two representations. We implemented both the Ordinary pipe-dream and bumpless pipe-dream representations in Python. We will cover bumpless pipe-dreams.

Definition. *A bumpless pipe dream is a $n \times n$ array of tiles of the following form:*



such that the tiles form strands that move from the right side of the $n \times n$ array to the bottom side.

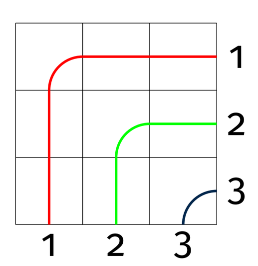


Figure 1: *Bumpless pipedreams for* 123

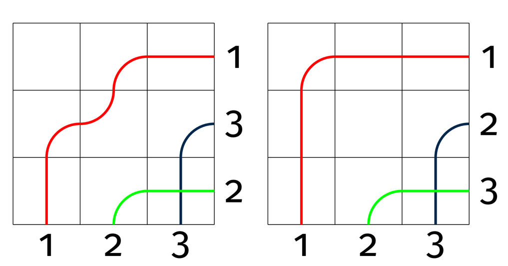


Figure 2: *Bumpless pipedreams for* 132

Each bumpless pipe dream:

- Has an associated permutation σ.
- Can be reduced or not reduced.
- Has a monomial weight.

The associated permutation can be found as follows:

1. Write $1 \dots n$ on each column at the bottom of the pipedream.
2. Trace these numbers to the right side of the pipedream, following the pipes.
3. Read from top down, where the top row is $\sigma(1)$ and the bottom row is $\sigma(n)$.

Definition. *A bumpless pipedream is reduced if and only if all strands cross at most once.*

Example 1. *The leftmost bumpless pipe dream is reduced, while the rightmost pipe dream is not reduced since two pipes cross more than once.*

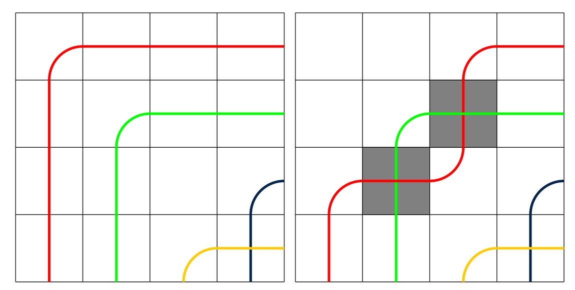


Figure 3: *Reduced vs non-reduced pipe dreams*

The monomial weight is $\prod x_i^{e_i}$, where e_i is the number of blank tiles in the ith row.

Theorem 1

A Schubert polynomial $\mathfrak{S}_\sigma$ is the sum of all monomial weights of all reduced bumpless pipe-dreams with the permutation representation σ.

Example 2. *The bumpless pipe-dreams on the left are all reduced bumpless pipe-dreams for* 3142.

- *Monomial weight for the left is $x_1^2x_3$*
- *Monomial weight for the right is $x_1^2x_2$*
- *Schubert polynomial: $\mathfrak{S}_{3142} = x_1^2x_3 + x_1^2x_2$*

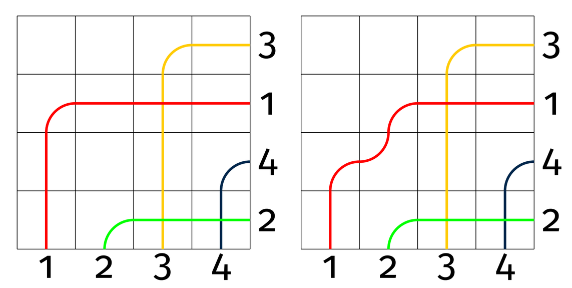


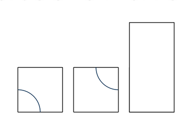
Figure 4: *Bumpless pipe dreams for* 3142

Results: Quantum Bumpless Pipe Dreams

Quantum Bumpless Pipe Dreams

We came up with a construction called *quantum bumpless pipe dreams* which generalize bumpless pipe dreams and give rise to quantum Schubert polynomial.

Definition: A quantum bumpless pipe dream is a $n \times n$ array of tiles that extends a bumpless pipe dream to **also** include tiles of the following form:



Note that the last tile is a domino tile that occupies two blank squares that are vertically adjacent.

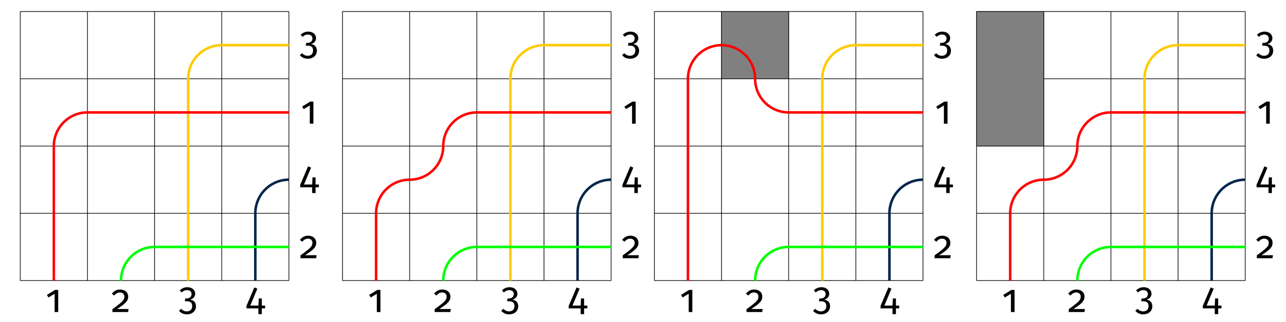


Figure 5: *Quantum bumpless pipe dreams for* 3142.

$$\mathfrak{S}^q_{3142} = x_1^2x_3 + x_1^2x_2 - q_1x_3 + q_1x_1$$

For a quantum bumpless pipe-dream, the strands move from the right to the bottom, during which they can move downward, upward or leftward. The monomial weights for a quantum bumpless pipe-dream can be calculated as the product of the following:

- An empty tile □ on row i contributes x_i
- A domino tile starting on row i contributes q_i
- A cross tile ⊞ on row i where the vertical strand moves upwards contributes q_i
- A southwest elbow ◺ on row i contributes $-q_i$
- A vertical tile ◫ on row i where the strand is moving upward contributes $-q_i$

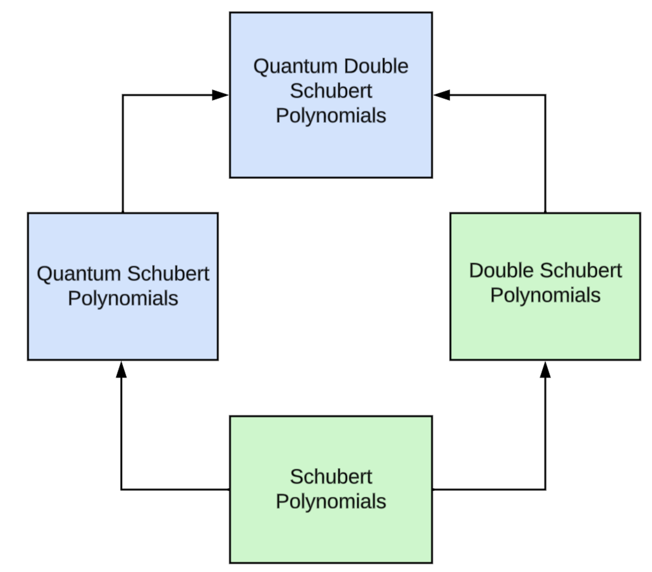
Theorem 2

The quantum Schubert polynomial for a permutation σ is equal to the sum of the weight of all quantum bumpless pipe dream for σ.

Double Schubert Polynomial

- A different generalization of Schubert polynomials
- A summation of binomials in terms of x_i and y_j rather than monomials

Quantum Double Schubert Polynomials are a generalization of both double and quantum Schubert polynomials



Our proof for this theorem is a bijective proof for a certain recurrence relation. This construction also generalizes naturally to quantum double bumpless pipe dreams by replacing the weight of an empty tile by $x_i - y_j$ where i is its row and j is its column. We also implemented the quantum bumpless pipe-dreams in a Python class (using Sage).

Future Directions

Quantum Pipe Dreams from Classical Pipe Dreams

- A different pipe dream formulation for Schubert polynomials
- there is a bijection from Bumpless Pipe Dreams to Classic Pipe Dreams
- Is it possible to construct a Quantum Ordinary Pipe Dream?

References

[1] Sergey Fomin, Sergei Gelfand, and Alexander Postnikov. Quantum schubert polynomials. *Journal of the American Mathematical Society*, 1997

[2] Nantel Bergeron and Sara Billey. Rc-graphs and schubert polynomials. *Experimental Mathematics*, 1993.