Generalizations of Newton's Method

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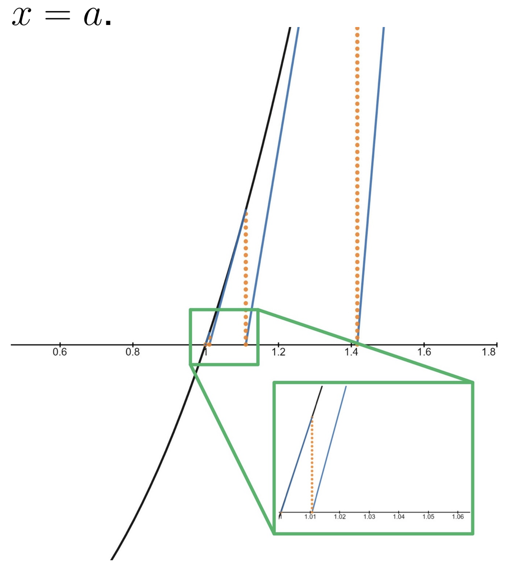
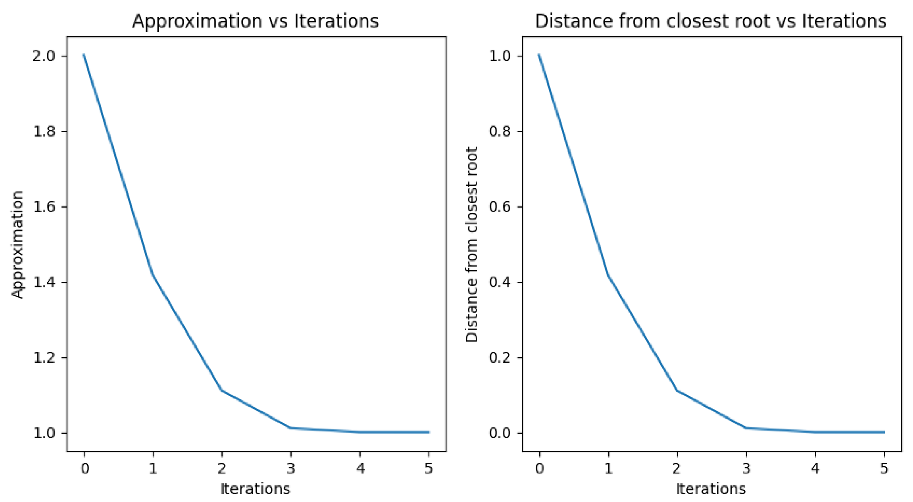
Introduction: What is Newton's Method

Newton's method is a numerical approach to gradually finding the solution of $f(x)$, using tangent lines to a curve as its foundation; it assumes $f'(a)$ is a slope of the tangent line at $x = a$.



1. Begin with an initial guess x_0
2. Draw tangent line at $(x_0, f(x_0))$
3. Name the intersection of the tangent line and the x-axis as x_1
4. Iterate the method until $x_n \approx x_{n+1}$

So after iterating the method, we get the root of the function $x^3 - 1$ as below. The figure below is the analysis of Newton's method, showing the convergence of the method to the root of the original function.



Thus, the formula of the classical Newton's Method is written below.

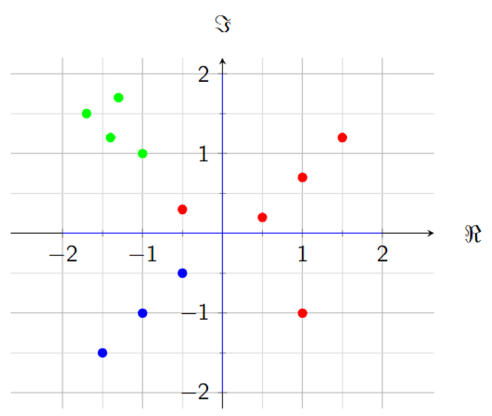
Newton's Method

Definition. *Suppose f is a differentiable function and $f'(x) \neq 0$. The Newton iteration formula associated to f is the formula*

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad n \geq 0.$$

Newton's Method over Complex Numbers

By using the formula, we can use Complex Numbers in Newton's Method.



1. Choose an arbitrary $c \in \mathbb{C}$
2. Apply Newton's Method for each c
3. Color the corresponding c with the specific color; color as the same if the c converges to the same root, and color as black if c does not converge
4. Iterate the method for each c color.

Iterating this method and when we get lots of colored dots on the plane, we then get a special, interesting figure, called **Newton's Fractal**.

Newton Fractals

- Newton fractals are the boundary of Newton's method in the complex plane.
- The boundary can be thought of as the points neighboring points that converge to a different root
- By coloring points by root converged to and shading by iterations required to converge, we can create amazing images!

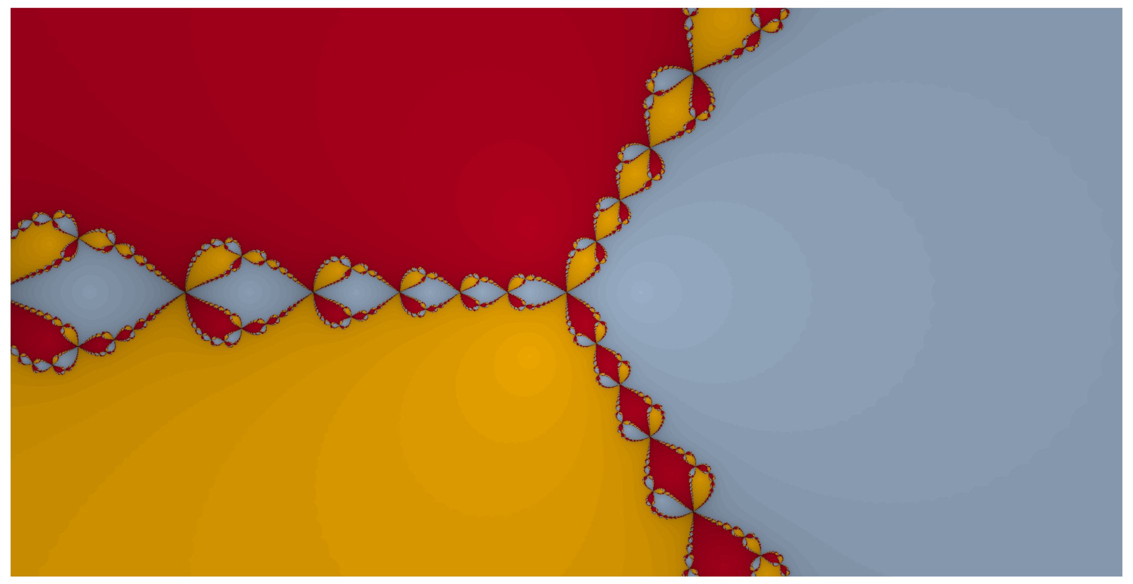


Figure 1: Newton fractal of $z^3 - 1$

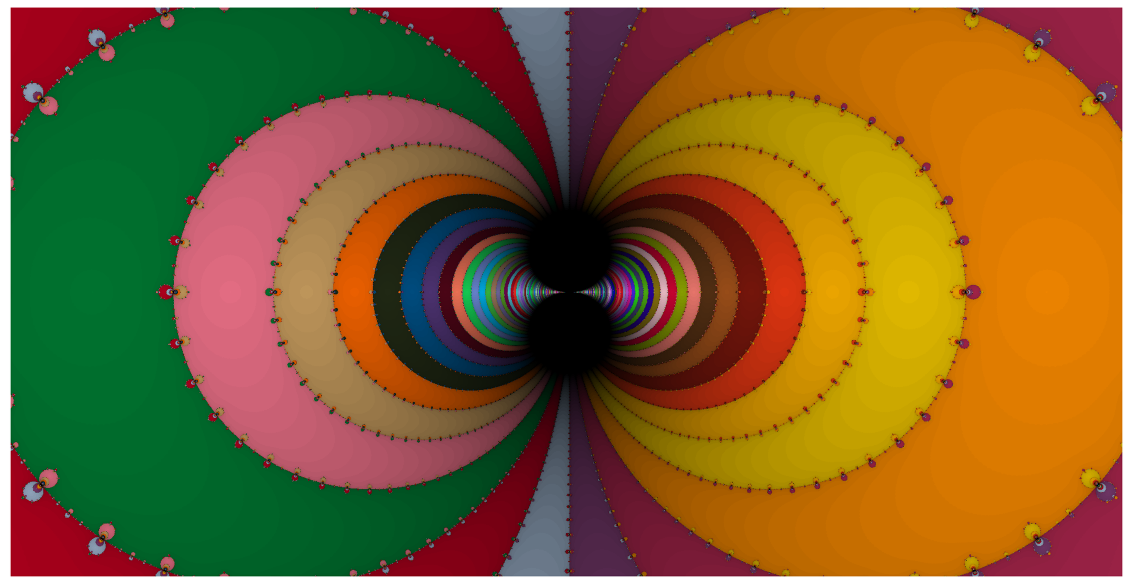
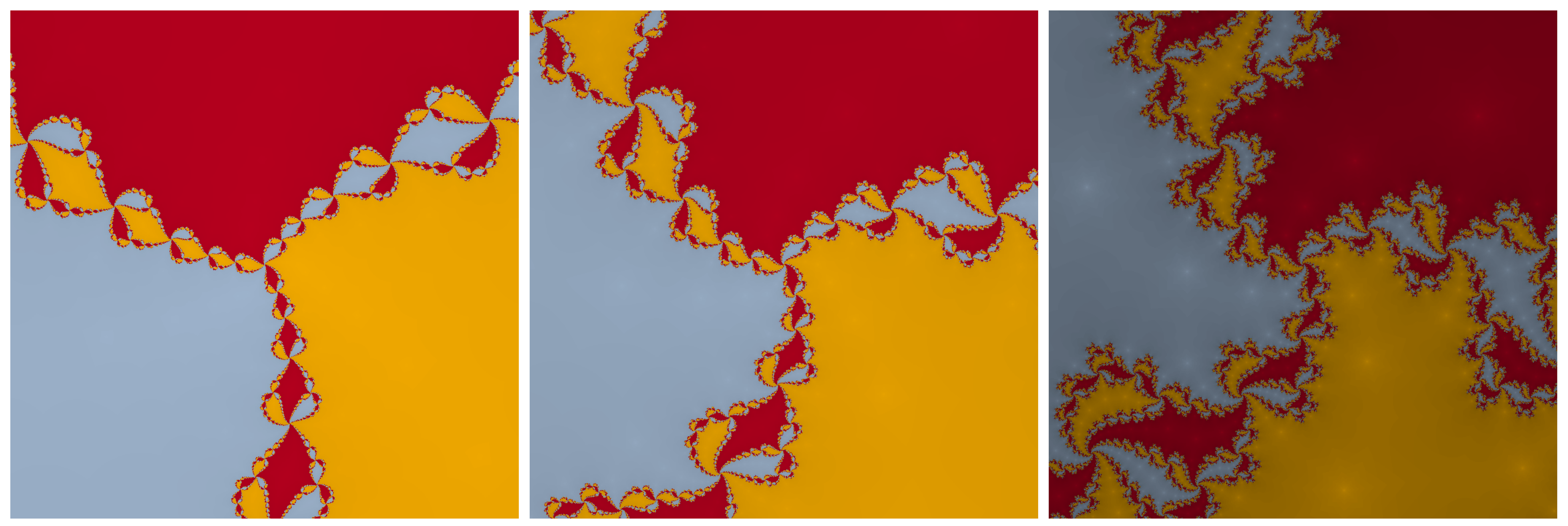


Figure 2: Newton fractal of $sin(z)$

We can generalize Newton's method by adding a constant to our iteration equation.

$$z_{n+1} = z_n - a\frac{f(z_n)}{f'(z_n)}$$

- Increasing the real part of a tends to make fractals more "spiky".
- Increasing the imaginary part of a tends to have a "swirling" effect.



$z^3 - 1$: $a = 1.2 + 0.2i$ | $z^3 - 1$: $a = 1.4 + 0.4i$ | $z^3 - 1$: $a = 1.6 + 0.6i$

Newton Graphs

We can also explore Newton's Method over sets with a finite number of elements. Our prototypical example is the integers mod p, $\mathbb{Z}_p = \{0, 1, ..., p-1\}$, where p is a prime.

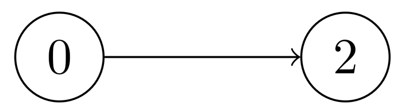
Let's look at an example of how we can use Newton's method on $f(x) = x^3 - 2x + 1$ on the finite field $\mathbb{Z}_3 \cup \{\infty\} = \{0, 1, 2, \infty\}$.

$$x - \frac{f(x)}{f'(x)} = x - \frac{x^3 - 2x + 1}{3x^2 - 2}$$

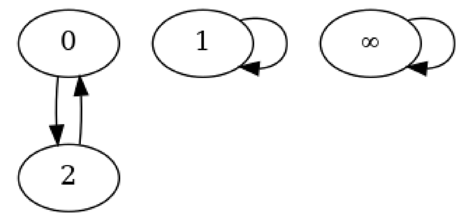
We can input elements from $\mathbb{Z}_3$ into our Newton map and record their outputs in a directed graph. For example, with 0:

$$0 - \frac{1}{-2} = 2$$

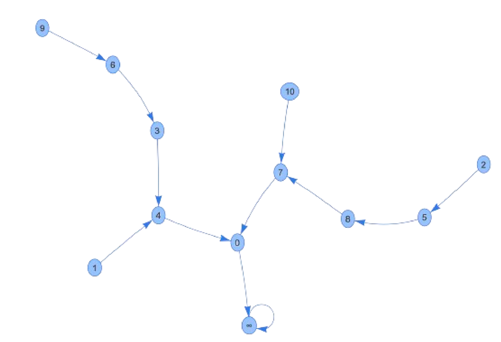
and so we draw an edge from 0 to 2.



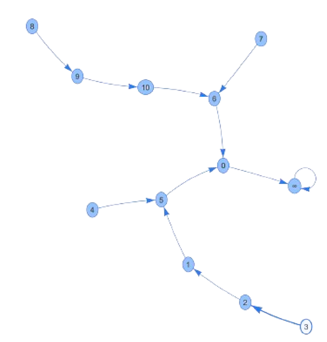
We can continue to do the same for the other elements.



Theorem: Suppose $\mathbb{Z}_p$ is a finite field where p is a prime. Then, $f(x) = x^n - q_1$ and $g(x) = x^n - q_2$ in $Z_p[x]$ have isomorphic Newton graphs if and only if $\exists c \in \mathbb{Z}_p$ such that $c \neq 0$ and $c^n q_1 = q_2$.



Newton graph of $f(x) = z^4 - 2$:



Newton graph of $g(x) = z^4 - 6$:

References

[1] Devaney. *A first course in chaotic dynamical systems: theory and experiment.* Journal. *American Institute of Physics.* 1993.

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[3] M.L. Sahari, I. Djellit *Fractal newton basins.* Journal. *Discrete Dynamics in Nature and Society.* 2006.

[4] Illinois Geometry Lab. IGL Poster Template. *University of Illinois at Urbana-Champaign Department of Mathematics,* 2017.

Newton fractal visualizer (ryanjvig.github.io/fractal.html)

Newton graph visualizer (ryanjvig.github.io/newtongraph.html)

Scan the QR codes above to check out our Newton fractal and Newton graph visualizers!