When Trading Distorts Prices: Feedback, Negative Prices, and Nonlinear PDEs

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When trading distorts prices

- Groups of traders pursuing similar objectives or facing similar constraints can collectively affect market prices.
- Trading feedback can exacerbate volatility and create correlations between markets.
- Examples include automated program trading in the 1987 crash and the financialization of commodity markets.
- Trading decisions affect prices, which in turn influence subsequent trading decisions.
- Market clearing and optimal trading lead to fixed-point problems, often characterized by nonlinear PDEs.

Dynamic option hedging: 1987 and beyond

- Program traders use option-hedging strategies, continuously adjusting positions in the underlying asset.
- Their aggregate demand affects the asset price and therefore changes the hedge that they must hold.
- Reference traders and program traders interact through a market-clearing condition.
- The option delta is determined by the derivative price and is not known *a priori* under feedback.
- Sircar–Papanicolaou (1998) derive a nonlinear pricing equation that reduces to Black–Scholes without feedback.

Sircar–Papanicolaou (1998): market clearing

- Reference traders demand $D(x, y)$ shares, given stock price x and stochastic income y.
- Program traders hedge ζ identical options; their demand per option is $\Phi(t, x)$ shares.
- With total shares outstanding S_0 and $\rho = \zeta / S_0$, normalized market clearing is

$$\boxed{D(X_t, Y_t) + \rho \Phi(t, X_t) = 1.}$$

- Assume $D_x < 0$, $D_y > 0$ and reference income $dY_t = \mu_Y \, dt + \eta_Y \, dW_t$.
- The observed price X_t adjusts to maintain market clearing after income or hedge-demand shocks.

Market clearing determines stock-price volatility

- Write $dX_t = \alpha_t\, dt + \beta_t\, dW_t$. Apply Itô's formula to $D(X_t, Y_t) + \rho\Phi(t, X_t) = 1$.
- Matching the dW_t coefficients gives

$$(D_x + \rho\Phi_x)\beta_t + D_y\eta_Y = 0.$$

$$\boxed{\beta_t = -\frac{D_y\eta_Y}{D_x + \rho\Phi_x}.}$$

- Without program trading ($\rho = 0$), $\beta_t^{(0)} = -D_y\eta_Y / D_x$.
- A positive slope Φ_x offsets the negative reference-demand slope and can amplify price fluctuations.

Replication feeds option gamma into volatility

- Let $C(t,x)$ price the hedged option. By Itô's formula, its diffusion coefficient is $C_x\beta_t$.
- A self-financing replicating portfolio holding Φ shares has diffusion coefficient $\Phi\beta_t$.

$$\boxed{\Phi = C_x, \qquad \Phi_x = C_{xx}.}$$

$$\boxed{\beta_t = -\frac{D_y \eta_Y}{D_x + \rho C_{xx}}.}$$

- Matching drift with the riskless bank account (r) gives

$$C_t + \tfrac{1}{2}\beta_t^2 C_{xx} + r(xC_x - C) = 0, \quad C(T,x) = h(x).$$

- The pricing equation is now nonlinear: volatility depends on the unknown gamma C_{xx}.

Choose reference demand to recover Black–Scholes

- Suppose $dY_t = \mu_1 Y_t\,dt + \eta_1 Y_t\,dW_t$; choose reference demand $D(x,y) = ay^\gamma/x$.
- In the absence of hedging demand, $-\eta_1 y D_y/D_x = \gamma\eta_1 x = \sigma x$; take $\sigma = \gamma\eta_1$.
- With program traders, market clearing and delta hedging imply

$$aY_t^\gamma = X_t(1-\rho C_x).$$

$$\boxed{\beta_t = \sigma X_t \frac{1-\rho C_x}{1-\rho C_x - \rho X_t C_{xx}}.}$$

- Substitute into replication to obtain the nonlinear Black–Scholes PDE on the next slide.

Nonlinear Black–Scholes equation: hedging feedback

- For a representative linear-demand specification, the derivative value $C(t,x)$ solves a nonlinear Black–Scholes equation.

$$\boxed{C_t + \frac{1}{2}\left[\frac{1-\rho C_x}{1-\rho C_x - \rho x C_{xx}}\right]^2 \sigma^2 x^2 C_{xx} + r(xC_x - C) = 0.}$$

- The feedback parameter ρ measures the influence of program traders on market clearing.
- The effective diffusion coefficient depends on both the option delta C_x and gamma C_{xx}.
- When $\rho = 0$, the classical linear Black–Scholes equation is recovered.
- The analysis predicts increased volatility under dynamic hedging feedback.

Financialization of commodity markets

- Commodity markets traditionally connect producers and industrial users through supply and demand.
- During the 2000s, commodity index funds and other financial participants became increasingly influential.
- Trading is motivated by diversification, speculation, and portfolio strategies, rather than direct commodity use.
- Tang and Xiong report increased co-movement of commodity prices with broader financial markets.

ie. Correlations between the commodities and the S&P 500: 1990-20
004-11.

Commodity, correlated to S&P	non-indexed period	indexed perio
CBOT Corn Futures	0.0099	0.2231
CSCE Cocoa Futures	-0.0182	0.1473
NYMEX Crude Oil Futures	-0.0546	0.3287
NYCE Cotton Futures	0.0116	0.1511
CSCE Coffee Futures	0.0290	0.1928
NYMEX Natural Gas Futures	-0.0092	0.1035
CSCE Sugar No. 11 Futures	-0.0082	0.1499
CBOT Soybean Futures	0.0124	0.2074
COMEX Silver Futures	-0.0516	0.1848
CBOT Wheat Futures	0.0005	0.1943

Commodity reference demand and market clearing

- Reference traders have stochastic income I_t and trade the commodity for industrial or hedging purposes.
- With fixed supply A, a simple reference demand is $D(Y_t, I_t) = I_t^\lambda / Y_t$, $\lambda > 0$.
- Without financial traders, market clearing gives the fundamental commodity price $Y_t^{(0)} = I_t^\lambda / A$.
- Adding portfolio optimizers with commodity investment fraction $\hat{\theta}_t$ changes market clearing:

$$\boxed{D(Y_t, I_t) + \widetilde{\varepsilon} \frac{\hat{\theta}_t X_t}{Y_t} = A.}$$

- Their collective demand shifts the commodity price away from the reference-price dynamics.

Portfolio optimizers: a feedback fixed point

- Investors allocate wealth among a money-market account, a stock index, and the commodity.
- With power utility, they choose stock and commodity fractions to maximize expected terminal utility.
- The price Y_t is determined by market clearing, but the optimal fractions depend on the dynamics of Y_t.
- At each stage, agents optimize under a proposed price process; aggregate trading then updates that price process.
- The mathematical question is whether the resulting sequence of impact stages has a fixed point.

Reference commodity dynamics: exponential OU

- With supply A and user demand $D(Y_t, I_t) = I_t^\lambda / Y_t$, market clearing without financial traders gives

$$\boxed{Y_t^{(0)} = \frac{I_t^\lambda}{A}.}$$

- For log-OU income I_t, the reference commodity price is also log-OU (Schwartz-type model):

$$\boxed{dY_t^{(0)} = a(m - \log Y_t^{(0)})Y_t^{(0)}dt + \lambda b Y_t^{(0)} dW_t^c.}$$

- The Brownian motion W^c represents commodity fundamentals; initially there is no exposure to stock-index shocks.
- Here b denotes the income-shock volatility, so the reference commodity volatility is λb.

Financial traders: stock index and wealth dynamics

- A portfolio optimizer invests fractions $\widehat{\pi}_t$ in a stock index S_t, $\widehat{\theta}_t$ in the commodity Y_t, and the remainder in cash earning r.

$$\boxed{dS_t = \mu S_t\, dt + \sigma S_t\, dW_t^s, \quad d\langle W^c, W^s\rangle_t = 0.}$$

- For investor wealth X_t, the self-financing budget equation is

$$\boxed{\frac{dX_t}{X_t} = \widehat{\pi}_t \frac{dS_t}{S_t} + \widehat{\theta}_t \frac{dY_t}{Y_t} + r(1 - \widehat{\pi}_t - \widehat{\theta}_t)dt.}$$

- Investors maximize $\mathbb{E}[U(X_T)]$ with $U(x) = x^{1-\gamma}/(1-\gamma)$; positions may be long or short.

Commodity financialization: equilibrium fixed point

- Aggregate commodity investment $\widehat{\theta}_t X_t$ must clear the market together with reference demand:

$$\boxed{\frac{I_t^\lambda}{Y_t} + \widetilde{\varepsilon}\frac{\widehat{\theta}_t X_t}{Y_t} = A.}$$

$$\boxed{Y_t = \frac{I_t^\lambda}{A} + \varepsilon \widehat{\theta}_t X_t, \quad \varepsilon = \widetilde{\varepsilon}/A.}$$

- But the optimizing fraction $\widehat{\theta}_t$ depends on the drift, volatility and correlation of Y_t.
- Those price dynamics, in turn, are determined by the optimizer's own aggregate demand.
- The model is therefore a fixed point between portfolio optimization and market clearing.

Financialization: volatility and correlation effects

- The reference commodity price has volatility $\eta^{(0)} = \lambda b$ and no correlation with the stock.
- The successive stages of optimization and market clearing change both volatility and correlation.

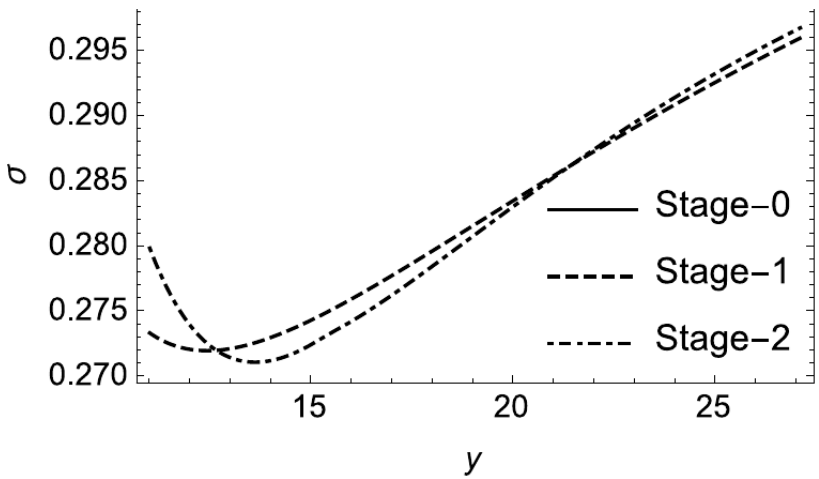


Figure: Volatility of the commodity price.

Commodity-price volatility

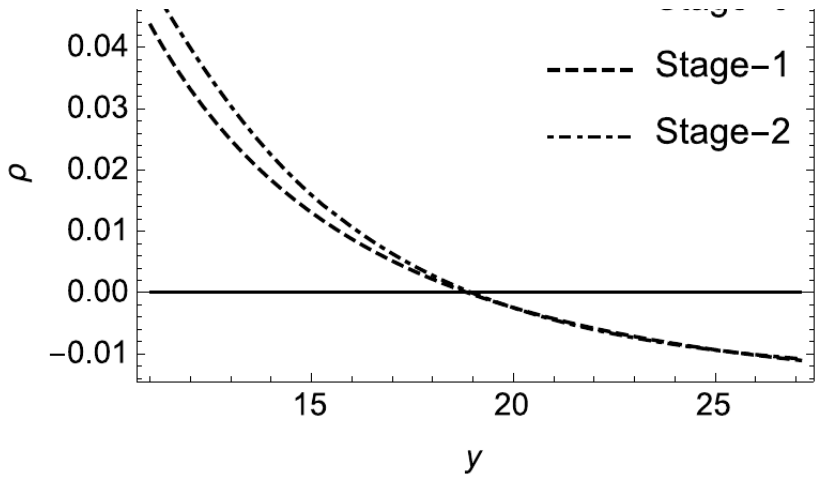


Figure: Correlation between the commodity and stock.

Stock–commodity correlation

- In these illustrations, the correlation switches sign as the optimal commodity position switches from long to short.

Merton's portfolio optimization problem

- The stock follows geometric Brownian motion $dS_t = \mu S_t dt + \sigma S_t dW_t$.
- An investor holds π_t dollars in stock; at zero interest, $dX_t = \pi_t \, dS_t / S_t$.
- The objective is $M(t,x) = \sup_\pi \mathbb{E}[U(X_T) \mid X_t = x]$, for an increasing concave utility U.
- The HJB equation and classical power-utility optimizer are

$$M_t - \frac{1}{2}\frac{\mu^2}{\sigma^2}\frac{M_x^2}{M_{xx}} = 0, \qquad \pi^* = \frac{\mu}{\gamma\sigma^2} X.$$

- For power, logarithmic, and exponential utilities, explicit solutions are available.

Black's PDE for the optimal portfolio

- Before introducing price impact, Merton's HJB provides a nonlinear PDE for the value function $M(t,x)$.
- Write $\lambda = \mu/\sigma$ and define the risk tolerance $R(t,x) = -M_x/M_{xx}$.
- **Black's equation (1968):** the risk tolerance satisfies a nonlinear fast-diffusion equation:

$$\boxed{R_t + \frac{1}{2}\lambda^2 R^2 R_{xx} = 0, \qquad R(T,x) = -\frac{U'(x)}{U''(x)}.}$$

- The optimal investment is $\pi^* = (\mu/\sigma^2)R$; equivalently $\pi^*_t + \frac{1}{2}\sigma^2(\pi^*)^2\pi^*_{xx} = 0$.
- For power utility, $R = x/\gamma$ and $\pi^* = \mu x/(\gamma\sigma^2)$. Feedback makes μ, σ endogenous.

Impact of portfolio optimizers

- Now the fundamental price S_t is perturbed by a continuum of portfolio optimizers.
- Aggregate demand π_t and market price P_t satisfy the market-clearing relation

$$\boxed{\frac{S_t}{P_t} + \varepsilon \frac{\pi_t}{P_t} = 1 \quad \Longrightarrow \quad P_t = S_t + \varepsilon \pi_t.}$$

- Each investor is small, but their aggregate demand has an effect on the market price.
- Investors optimize against $dP_t/P_t = \mu_p dt + \sigma_p dW_t$, where μ_p and σ_p are endogenous.
- Optimization and market clearing must be satisfied simultaneously.

Coupled nonlinear equilibrium equations

- With power utility, homogeneity gives an individual optimal investment linear in individual wealth.
- Writing $\pi = \pi(t, x, s)$ and $H = H(t, x, s)$, market equilibrium is characterized by a coupled PDE system.

$$H_t + \mathcal{L}_{x,s} H + \frac{1-\gamma}{\gamma} \lambda_p (\pi \sigma_p H_x + s \sigma_s H_s) + \frac{1-\gamma}{2\gamma^2} \lambda_p^2 H = 0.$$

$$\pi = x \left(\frac{\mu_p}{\gamma \sigma_p^2} + \pi \frac{H_x}{H} + \frac{s \sigma_s}{\sigma_p} \frac{H_s}{H} \right).$$

- Both μ_p and σ_p depend on the strategy π and its derivatives through market clearing.
- The state dimension reduces by the homogeneous variable $y = x/s$.

Terminal conditions and equilibrium selection

- Terminal utility determines $H(T, x, s) = 1$, but does not determine the aggregate terminal strategy $\pi(T, x, s)$.
- The stages approach can select different terminal conditions depending on its initialization.
- Case I: assume no feedback at the terminal time, using the classical Merton strategy.

$$\pi(T, x, s) = \frac{\mu_s}{\gamma \sigma_s^2} x.$$

- Case II: traders exit the risky asset at the terminal time, $\pi(T, x, s) = 0$.
- Different terminal prescriptions produce markedly different numerical equilibrium price dynamics.

Portfolio feedback: two terminal prescriptions

- The same parameter values are used for the two computations: $T = 0.5$, $\mu_s = 0.25$, $\sigma_s = 0.35$, $\gamma = 0.8$, $\varepsilon = 0.4$.
- Case I imposes the Merton terminal strategy; Case II imposes zero terminal demand.

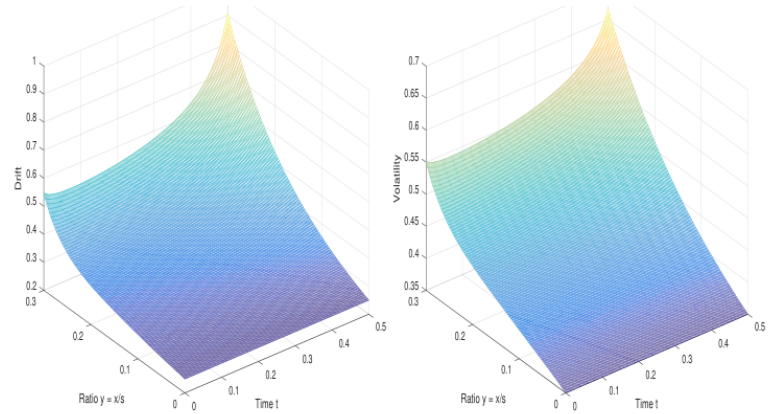


Figure: The drift and volatility of market price under Merton terminal

Case I: Merton terminal strategy

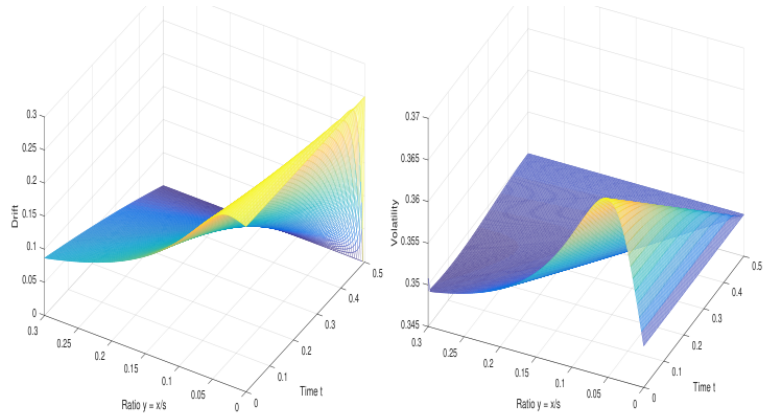


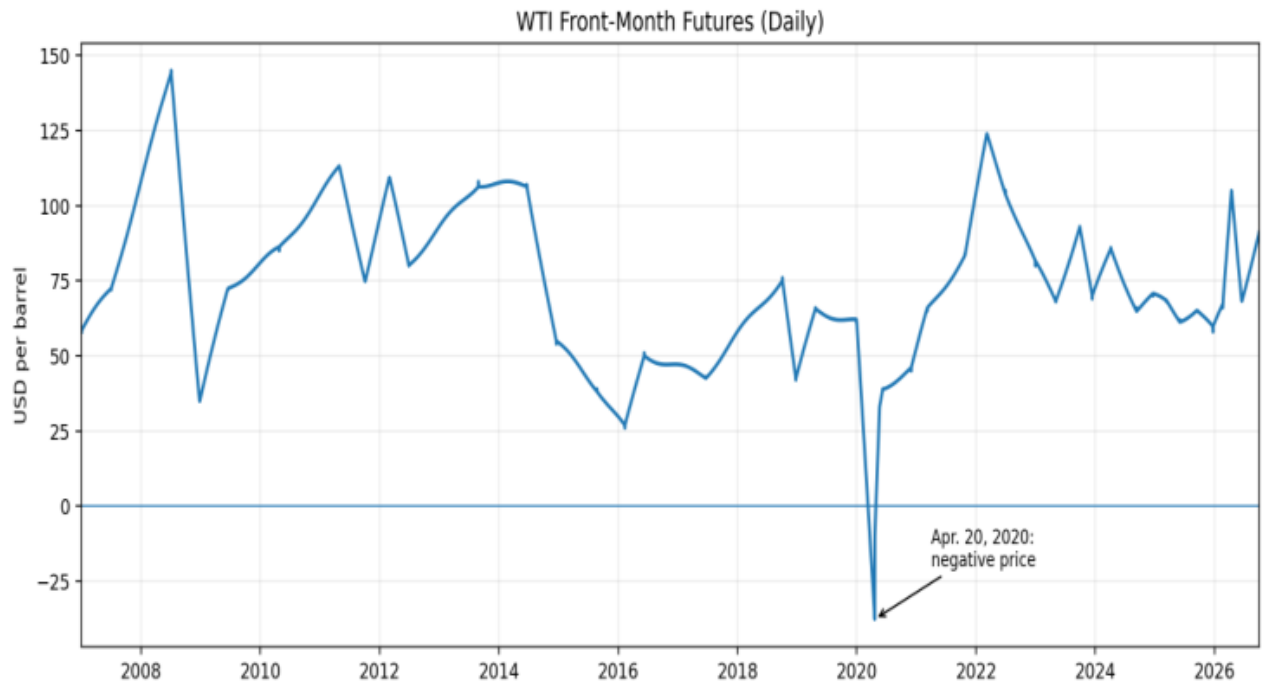
Figure: The drift and volatility of market price under zero terminal

Case II: zero terminal strategy

- The computed drift and volatility surfaces demonstrate sensitivity to the terminal-condition selection.

WTI oil front-month futures: 2007–2026

- The April 2020 event stands out in the historical record of front-month WTI futures prices.
- The May 2020 contract settled below zero on April 20, one trading day before expiration.



Illustrative clean line version of the front-month WTI futures history through Sep. 30, 2026.

The April 20, 2020 oil collapse

- The May 2020 WTI contract, on its penultimate trading day, initially traded at around \$17 per barrel.
- The contract settled at $-\$37.63$ per barrel; this negative-price event was unforeseen at the time.
- Physically settled futures require delivery at maturity; financial traders without delivery capability must close or roll positions.
- The COVID-19 pandemic sharply reduced oil demand and placed pressure on physical storage capacity.
- Cushing inventories reached 59.7 million barrels on April 17; open interest was 108,593 lots.
- Can forced liquidation by traders unable to take delivery amplify the price decline?

Two types of traders

- Licensed traders: producers, merchants, and commodity trading firms able to make or take physical delivery.
- Unlicensed traders: financial participants who trade futures but cannot participate in physical delivery.
- Licensed traders may hold the front-month contract to maturity.
- Unlicensed traders must close or roll positions by an earlier time $T' < T$.
- An imbalance between unlicensed long and short positions determines the direction of forced trading pressure.

Open interest and imbalance parameter

- At $t < T$, write licensed and unlicensed long/short positions as $\ell_t^L, \ell_t^U, s_t^L, s_t^U$.
- Total open interest has equal long and short totals:

$$\mathrm{OI}_t = \ell_t^L + \ell_t^U = s_t^L + s_t^U.$$

- Let $T' < T$ be the time by which unlicensed positions must be closed.
- The fraction of total open interest represented by net unlicensed exposure is

$$\boxed{\varepsilon = \frac{|\ell_{T'}^U - s_{T'}^U|}{\mathrm{OI}_{T'}}, \qquad 0 \le \varepsilon < 1.}$$

- Excess unlicensed longs yield downward pressure; excess shorts yield upward pressure.

Forced long rollover and price distortion

- Case 1: unlicensed long positions outnumber unlicensed short positions.
- Excess unlicensed longs close positions against licensed shorts before delivery.
- Observed price is a volume-weighted average of a licensed price and a rollover value H_{t,T,T_2}.

$$F^{\text{obs}}_{t,T} = (1-\varepsilon)F^{\text{lic}}_{t,T} + \varepsilon H_{t,T,T_2}.$$

- The long-side roll option $R^{(+)}_t$ measures the cost of being unable to take delivery.
- With $H_{t,T,T_2} = F^{\text{lic}}_{t,T} - R^{(+)}_t$, observed price lies below licensed price.

$$\boxed{F^{\text{obs}}_{t,T} = F^{\text{lic}}_{t,T} - \varepsilon R^{(+)}_t.}$$

Roll-option payoff and the feedback relation

- Let $T_2 > T$ be the next contract maturity, with $D = e^{-r(T_2 - T)}$.
- At expiration, the long-side rollover cost has payoff

$$R_T^{(+)} = \left(D F_{T,T_2}^{\text{obs}} - F_{T,T}^{\text{obs}}\right)^+.$$

- The option value is $R_t^{(+)} = \mathbb{E}^{\mathbb{Q}}[R_T^{(+)} \mid \mathcal{F}_t]$.
- The next-month contract is treated as undistorted over the relevant period.
- Combining price formation and valuation produces an implicit nonlinear relation:

$$\boxed{F_{t,T}^{\text{obs}} = F_{t,T}^{\text{lic}} - \varepsilon \mathbb{E}^{\mathbb{Q}}\left[(D F_{T,T_2}^{\text{obs}} - F_{T,T}^{\text{obs}})^+ \mid \mathcal{F}_t\right].}$$

Oil feedback: benchmark dynamics

Under the risk-neutral measure $\mathbb{Q}$, the licensed front-month price and the next-month futures price are modeled by correlated geometric Brownian motions:

$$\frac{dF^{\text{lic}}_{t,T}}{F^{\text{lic}}_{t,T}} = \sigma_1(t)\, dW^1_t, \qquad \frac{dF^{\text{obs}}_{t,T_2}}{F^{\text{obs}}_{t,T_2}} = \sigma_2(t)\, dW^2_t,$$

$$d\langle W^1, W^2\rangle_t = \rho\, dt.$$

- Write $f_1 = F^{\text{obs}}_{t,T}$, $f_2 = F^{\text{obs}}_{t,T_2}$ and $R^{(+)}_t = V^+(t, f_1, f_2)$.
- The price-feedback identity is

$$\boxed{F^{\text{lic}}_{t,T} = f_1 + \varepsilon V^+(t, f_1, f_2).}$$

- Both $F^{\text{lic}}_{t,T}$ and $R^{(+)}_t$ are $\mathbb{Q}$-martingales under the forward-pricing convention; hence f_1 is also a martingale.

Derivation of the V^+ PDE: Itô's formula

As the observed front-month price is a continuous martingale, write

$$df_1 = \beta_t\, dW_t^1 + \gamma_t\, dW_t^2, \qquad df_2 = f_2\sigma_2(t)\, dW_t^2.$$

Applying Itô's formula to $V^+(t, f_1, f_2)$ gives

$$\begin{aligned} dV^+ = {} & V^+_{f_1}(\beta_t\, dW_t^1 + \gamma_t\, dW_t^2) + V^+_{f_2} f_2\sigma_2\, dW_t^2 \\ & + \Big[V_t^+ + \frac{1}{2}(\beta_t^2 + 2\rho\beta_t\gamma_t + \gamma_t^2)V^+_{f_1 f_1} \\ & + (\rho\beta_t + \gamma_t) f_2\sigma_2 V^+_{f_1 f_2} + \frac{1}{2} f_2^2\sigma_2^2 V^+_{f_2 f_2}\Big]\, dt. \end{aligned}$$

- Since V^+ is a martingale, its drift coefficient must vanish.
- To identify β_t, γ_t, differentiate the feedback identity.

Derivation of the V^+ PDE: coefficient matching

Differentiating $f_1 + \varepsilon V^+ = F^{\text{lic}}_{t,T}$ and matching dW^1_t, dW^2_t coefficients gives

$$\beta_t + \varepsilon V^+_{f_1} \beta_t = (f_1 + \varepsilon V^+)\sigma_1(t),$$
$$\gamma_t + \varepsilon\big(V^+_{f_1}\gamma_t + V^+_{f_2} f_2 \sigma_2(t)\big) = 0.$$

Thus, with $A^+ = 1 + \varepsilon V^+_{f_1}$,

$$\boxed{\beta_t = \frac{(f_1 + \varepsilon V^+)\sigma_1(t)}{A^+}, \qquad \gamma_t = -\frac{\varepsilon f_2 \sigma_2(t) V^+_{f_2}}{A^+}.}$$

- The diffusion of the observed front-month price depends on the roll-option value and its sensitivities.
- Substitute these expressions into the drift of Itô's formula and set it to zero.

The nonlinear feedback PDE for V^+

The martingale condition yields

$$\boxed{\begin{aligned} 0 = V_t^+ + \frac{1}{2} J^+ \sigma_1^2(t) f_1^2 V_{f_1 f_1}^+ \\ + K^+ \sigma_1(t)\sigma_2(t) f_1 f_2 V_{f_1 f_2}^+ + \frac{1}{2}\sigma_2^2(t) f_2^2 V_{f_2 f_2}^+ . \end{aligned}}$$

with terminal condition

$$\boxed{V^+(T, f_1, f_2) = (D f_2 - f_1)^+ .}$$

- The coefficients J^+ and K^+ are nonlinear functions of V^+ and its first derivatives.
- Consequently the option price and the observed-price diffusion must be determined self-consistently.

Feedback diffusion coefficients J^+ and K^+

Define the three quantities (with $\sigma_1(t) > 0$)

$$A^+ = 1 + \varepsilon V^+_{f_1},$$
$$B^+ = 1 + \varepsilon \frac{V^+}{f_1},$$
$$C^+ = \frac{f_2}{f_1}\frac{\sigma_2(t)}{\sigma_1(t)} V^+_{f_2}.$$

Then the nonlinear diffusion multipliers are

$$\boxed{J^+ = \frac{(B^+)^2 + \varepsilon^2 (C^+)^2 - 2\rho\varepsilon B^+ C^+}{(A^+)^2},}$$

$$\boxed{K^+ = \frac{\rho B^+ - \varepsilon C^+}{A^+}.}$$

- At $\varepsilon = 0$: $A^+ = B^+ = J^+ = 1$ and $K^+ = \rho$, recovering the standard exchange-option PDE.

The no-feedback Margrabe benchmark

- For $\varepsilon = 0$, the calendar exchange option is valued by the Black–Scholes–Margrabe formula.
- Writing $\xi = f_1/f_2$, the no-feedback price is $V^{(0)+}(t, f_1, f_2) = f_2 P(t, \xi)$.
- The put function solves the linear equation

$$P_t + \tfrac{1}{2}\sigma_M^2(t)\xi^2 P_{\xi\xi} = 0, \qquad P(T, \xi) = (D - \xi)^+.$$

$$\sigma_M^2(t) = \sigma_1^2(t) + \sigma_2^2(t) - 2\rho\sigma_1(t)\sigma_2(t).$$

- Its explicit formula is $P(t, \xi) = DN(-d_2) - \xi N(-d_1)$, with integrated Margrabe variance.
- This known solution will reappear in the full feedback model.

Homogeneity reduces the nonlinear PDE

- The payoff $(Df_2 - f_1)^+$ is homogeneous of degree one in the two futures prices.
- Seek the feedback value in the form

$$\boxed{V^+(t, f_1, f_2) = f_2 G^+(t, \xi), \qquad \xi = \frac{f_1}{f_2}.}$$

- Then $V^+_{f_1} = G^+_\xi$, while $V^+_{f_2} = G^+ - \xi G^+_\xi$.
- The terms in the two-dimensional pricing equation combine into a single second derivative $G^+_{\xi\xi}$.
- One obtains a nonlinear parabolic PDE in only one state variable.

Reduced nonlinear feedback PDE

- With $A^+ = 1 + \varepsilon G^+_\xi$ and $B^+ = 1 + \varepsilon G^+/\xi$, the reduced equation is

$$\boxed{G^+_t + \frac{1}{2}\left(\frac{B^+}{A^+}\right)^2 \sigma_M^2(t)\xi^2 G^+_{\xi\xi} = 0.}$$

$$G^+(T,\xi) = (D-\xi)^+.$$

- Equivalently, the effective diffusion coefficient contains $\left[(\xi + \varepsilon G^+)/(1 + \varepsilon G^+_\xi)\right]^2$.
- The nonlinearity is not a generic perturbation: it has a special fractional derivative structure.
- This structure admits a nonlinear change of variables that reduces the PDE to a linear equation.

Explicit solution: nonlinear change of variables

- Introduce the new spatial variable and transformed function

$$\boxed{y(t,\xi) = \xi + \varepsilon G^+(t,\xi), \qquad G^+(t,\xi) = H(t,y).}$$

- Under this change of variables, the nonlinear equation becomes the classical linear Margrabe PDE for H.
- The resulting solution is $H(t,y) = P(t,y)/(1-\varepsilon)$.
- Returning to the observed variable ξ gives a scalar implicit equation:

$$\boxed{G^+(t,\xi) = \frac{1}{1-\varepsilon} P\big(t, \xi + \varepsilon G^+(t,\xi)\big).}$$

- Since $-1 \le P_y \le 0$ and $0 \le \varepsilon < 1$, the scalar fixed-point equation has a unique solution.

How observed futures prices become negative

- The licensed futures price remains positive under the lognormal benchmark.
- The observed price is lower by the nonnegative feedback correction $-\varepsilon V^{+}$.
- Introduce $z = \xi + \varepsilon G^{+}(t, \xi)$; then the scalar equation implies

$$\xi = z - \frac{\varepsilon}{1-\varepsilon} P(t, z).$$

- As $z \downarrow 0$, $P(t, z) \to D$, and the admissible observed-price ratio extends through zero.

$$\boxed{\xi > -\frac{\varepsilon D}{1-\varepsilon}, \qquad f_1 > -\frac{\varepsilon D}{1-\varepsilon} f_2.}$$

- The lognormal fundamental benchmark can remain positive while the observed price turns negative.

Bachelier benchmark: arithmetic futures dynamics

- The lognormal benchmark keeps the licensed front-month futures price positive, even though feedback can make the observed price negative.
- For regimes such as April 2020, we may instead allow the licensed benchmark itself to cross zero.
- Under the risk-neutral measure, use *arithmetic Brownian motions*:

$$\boxed{\begin{aligned} dF^{\mathrm{lic}}_{t,T} &= \bar{\sigma}_1(t)\, dW^1_t, \\ dF^{\mathrm{obs}}_{t,T_2} &= \bar{\sigma}_2(t)\, dW^2_t, \qquad d\langle W^1, W^2\rangle_t = \rho\, dt. \end{aligned}}$$

- Scaling homogeneity is replaced by translation invariance: the relevant state variable is the *calendar spread*, not the price ratio.
- The no-feedback Margrabe formula is replaced by the corresponding *Bachelier* (normal) option formula.

Bachelier model: reduction to the calendar spread

- Set $s = Df_2 - f_1$, with $D = D(T, T_2)$, and seek a value function of the form

$$\boxed{V^+(t, f_1, f_2) = G_B^+(t, s), \qquad s = Df_2 - f_1.}$$

- The feedback identity $F_{t,T}^{\text{lic}} = f_1 + \varepsilon V^+$ gives the fundamental spread $s - \varepsilon G_B^+(t, s)$.
- Matching diffusion coefficients and imposing the martingale condition yields

$$\boxed{\begin{aligned}(G_B^+)_t + \frac{\Sigma_N^2(t)}{2[1 - \varepsilon(G_B^+)_s]^2}(G_B^+)_{ss} = 0,\\ G_B^+(T, s) = s^+,\end{aligned}}$$

$$\Sigma_N^2(t) = \bar{\sigma}_1^2(t) - 2\rho D\bar{\sigma}_1(t)\bar{\sigma}_2(t) + D^2\bar{\sigma}_2^2(t).$$

Bachelier pricing formula and explicit feedback solution

- When $\varepsilon = 0$, the reduced equation is a heat equation with normal (Bachelier) call payoff s^+.

$$\boxed{P_B(t,s) = s\,N\left(\frac{s}{v}\right) + v\,\varphi\left(\frac{s}{v}\right), \qquad v^2 = \int_t^T \Sigma_N^2(u)\,du.}$$

- With feedback, the nonlinear PDE has an implicit scalar solution analogous to the lognormal case:

$$\boxed{G_B^+(t,s) = \frac{1}{1-\varepsilon} P_B\left(t, s - \varepsilon G_B^+(t,s)\right).}$$

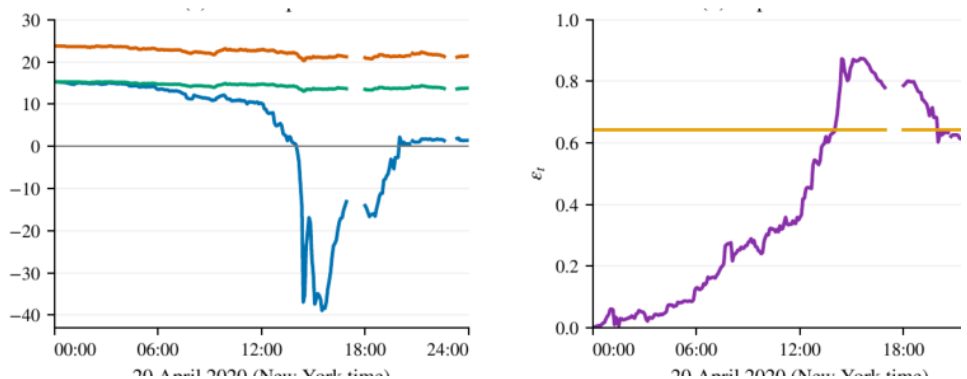
- The normal benchmark itself can cross zero; feedback adds a further distortion between licensed and observed futures prices.
- For $0 \le \varepsilon < 1$, the scalar equation has a unique nonnegative solution.

WTI settlement and the implied imbalance

- On April 20, 2020, official May and June WTI settlements were $-\$37.63$ and $\$20.43$ per barrel.
- The model's implied imbalance ε is inferred from the observed and benchmark futures prices.
- For the settlement inputs illustrated in the oil slides, the implied imbalance is approximately $\varepsilon = 0.87$.
- An intraday illustration follows the June contract as a positive benchmark for the licensed front-month price.

$$F^{\mathrm{lic}}_{t,T} = \lambda F^{\mathrm{obs}}_{t,T_2}, \qquad \lambda = \frac{\hat{F}_{t_0,T}}{F^{\mathrm{obs}}_{t_0,T_2}} \approx 0.64.$$

The May price falls sharply while the June price, and hence the assumed benchmark, remain positive.



Conclusions: nonlinear PDEs from trading feedback

- Dynamic hedging feedback leads to nonlinear Black–Scholes equations and can amplify volatility.
- Portfolio investment in commodity markets can generate additional volatility and correlations across assets.
- When trading strategies arise from stochastic optimization, feedback yields coupled nonlinear PDE systems and questions of terminal-condition selection.
- Delivery constraints in commodity futures produce a different feedback: observed prices can move far from positive benchmark prices.
- For the oil model, homogeneity and a nonlinear change of variables reduce the nonlinear PDE to a classical Margrabe equation and a unique scalar root.