Asymptotic Symmetries and the Soft Photon Theorem in Arbitrary Dimensions

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Michigan Brown Bag Seminar

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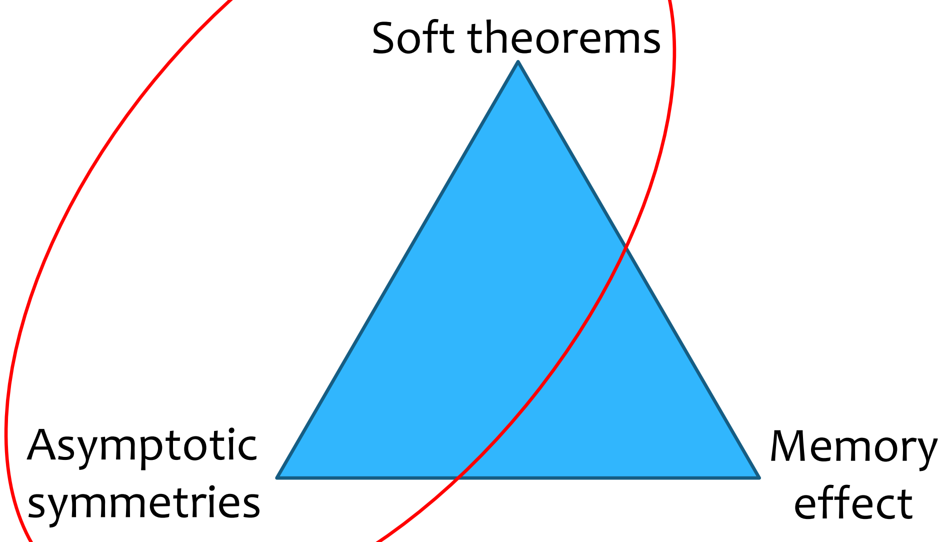
Collaboration

* Main focus on 1903.02608 (& 1903.03607), done in collaboration with P. Mitra
* Focuses on the soft theorem/asymptotic symmetry side of the triangle that Strominger introduced



History

* Relevant papers
  * 1407.3789 (TH, Mitra, Porfyriadis, Strominger): Massless QED in 4D
  * 1505.05346 (Campiglia, Laddha) and 1506.02906 (Kapec, Pate, Strominger) : Massive QED in 4D
  * 1412.2763 (Kapec, Lysov, Strominger): Massless QED in even dimensions

Goal of presentation

- **Leading soft photon theorem** – A theorem in QFT first discovered by studying Feynman diagrams
- **Asymptotic symmetry group of QED** – Large gauge transformations that leave the EOM invariant while changing the physical state
- We would like to demonstrate that the Ward identity associated to the **asymptotic symmetries** of QED is equivalent to the **leading soft photon theorem** in all dimensions.

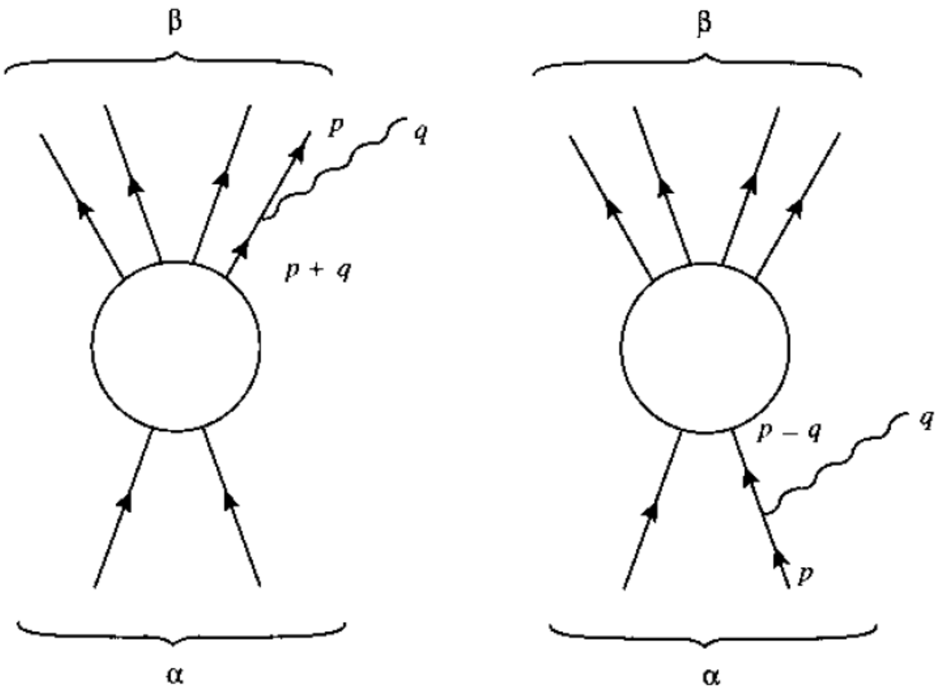
Outline of presentation

* Review the leading soft photon theorem and asymptotic symmetries
* Construct the charge generating asymptotic symmetries in (d+2)-dimensional QED
* Establish the Ward identity associated to the asymptotic symmetries
* Demonstrate how the Ward identity arises from the leading soft photon theorem in d+2 dimensions

Soft photon theorem

* Weinberg's soft photon theorem relates the matrix elements of a Feynman diagram with an external soft photon insertion to that of the same diagram without an external soft photon.

Soft photon theorem



From Weinberg, *The Quantum Theory of Fields, Vol. 1*

* Two ways to insert a soft photon
* In the diagrams we take $q \to 0$.

Leading soft photon theorem

* In equation form, assuming all particles are outgoing for simplicity,

$$\lim_{q\to 0}\mathcal{M}^{\pm}_{n+1}(q)=\lim_{q\to 0}\sum_{k=1}^{n}\frac{e_k p_k^{\mu}\varepsilon^{\pm}_{\mu}(q)}{p_k\cdot q}\mathcal{M}_n$$

* Universality suggests this theorem might arise from some symmetry.

Asymptotic symmetries

* Symmetries are one of the most useful tools in theoretical physics.
* One way to think of certain symmetries is in terms of a Ward identity, i.e. their charges commute with the S-matrix.
* Asymptotic symmetries are symmetries that act on the physical states in a nontrivial manner.

Asymptotic symmetries

* In many cases, asymptotic symmetries are local in spacetime coordinates.
* Sometimes called "large gauge symmetries" to distinguish from trivial gauge symmetries
* This definition of large gauge symmetries allows for topologically trivial symmetries as well.
* Large gauge symmetry physical if and only if it gives rise to nontrivial Ward identity

Remarkable equivalence

* Certainly some asymptotic symmetries are physical!
* Strong link established between many asymptotic symmetries and soft theorems – soft theorems are just Ward identities for asymptotic symmetries
* Examples:
  * Leading soft photon theorem = Ward identity for asymptotic symmetry in QED in 4D (TH, Mitra, Porfyriadis, Strominger)
  * Leading soft graviton theorem = Ward identity for supertranslations in 4D (TH, Mitra, Lysov, Strominger)

Remarkable equivalence

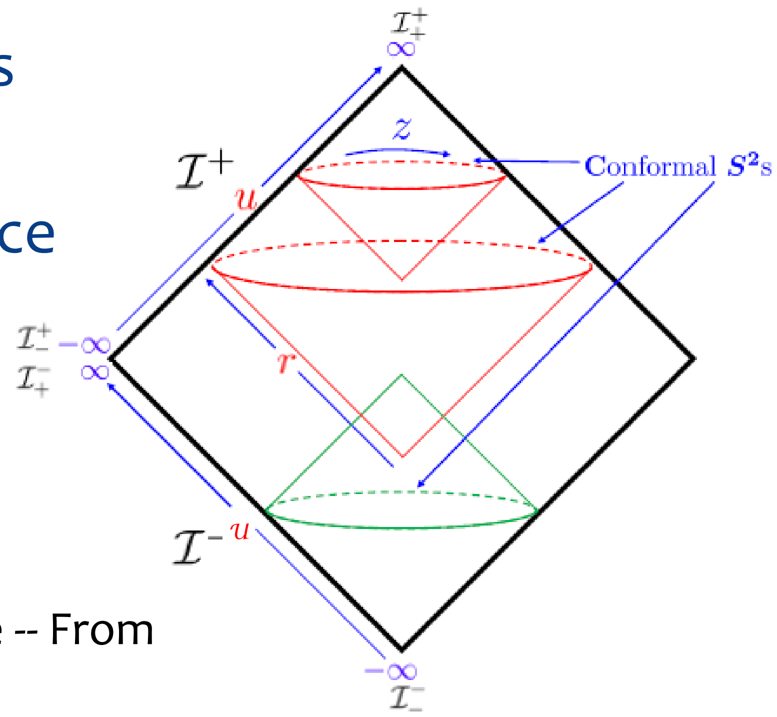
* This equivalence is remarkable!
  * Soft theorems were first studied in QED in 1937 by Bloch and Nordsieck, later by Low et al in 1958, then later extended by Weinberg to gravity in 1965.
  * Asymptotic symmetries appeared in the work of BMS in 1962, where they deduced the symmetry group for asymptotically flat spacetimes.
* One uses perturbative Feynman diagrams, the other uses asymptotic structures at null infinity.
* We attempt now to extend this equivalence to odd dimensions.

Qualitative differences

* Massless Green's function
  * Supported only on the lightcone in even dimensions
  * Supported on the entire interior of the lightcone in odd dimensions
* Asymptotics of gauge field
  * Analytic in even dimensions and admits Taylor expansion
  * Non-analytic in odd dimensions
* We still believe a connection between soft theorems and asymptotic symmetries.

Switching coordinate systems

- Easier to work in flat null coordinates in d+2 dimensions
- We choose our asymptotic boundary to be $\mathcal{I}^+$ and $\mathcal{I}^-$. Hence we only focus on massless particles.



In 4D spacetime -- From A. Strominger, arXiv:1312.2229 [hep-th]

Choosing our coordinates

* The (d+2)-dimensional metric is given by

$$ds^2 = \eta_{AB} dX^A dX^B = -du dr + r^2 \delta_{ab} dx^a dx^b$$

with

$$u = \frac{(X^0)^2 - X^a X_a - (X^{d+1})^2}{X^0 + X^{d+1}}, \qquad x^a = \frac{X^a}{X^0 + X^{d+1}}, \qquad r = X^0 + X^{d+1}$$

Equations of motion

* Maxwell's equation is

$$e^2 J_\nu = \nabla^\mu F_{\mu\nu}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

* There is a local symmetry under which

$$A_\mu \;\to\; A_\mu + \partial_\mu \varepsilon$$

Radiative and Coulombic modes

* We decompose general solution to Maxwell's equation as

$$F_{\mu\nu} = F_{\mu\nu}^{(R)} + F_{\mu\nu}^{(C)}$$

* Radiative field satisfies homogeneous Maxwell's equation, and Coulombic field is sourced by the matter current.

Asymptotic expansion at $\mathcal{I}^{\pm}$

- Near $\mathcal{I}^{\pm}$, the field strength obeys the fall-off conditions

$$F_{ur}^{(R\pm)} = O\left(|r|^{-\frac{d}{2}-1}\right) + O\left(|r|^{-d}\right), \qquad F_{ur}^{(C\pm)} = O\left(|r|^{-d}\right)$$

- The matter current also obeys

$$J_u = O\big(|r|^{-d}\big), \qquad J_a = O\big(|r|^{-d}\big), \qquad J_r = O\big(|r|^{-d-2}\big).$$

Leading constraint equation

* We write

$$F_{ur}^{(C\pm)}(u,r,x) = \sum_{n=0}^{\infty} \frac{F_{ur}^{(C\pm,d+n)}(u,x)}{|r|^{d+n}}$$

$$J_u(u,r,x) = \sum_{n=0}^{\infty} \frac{J_u^{(\pm,d+n)}(u,x)}{|r|^{d+n}}$$

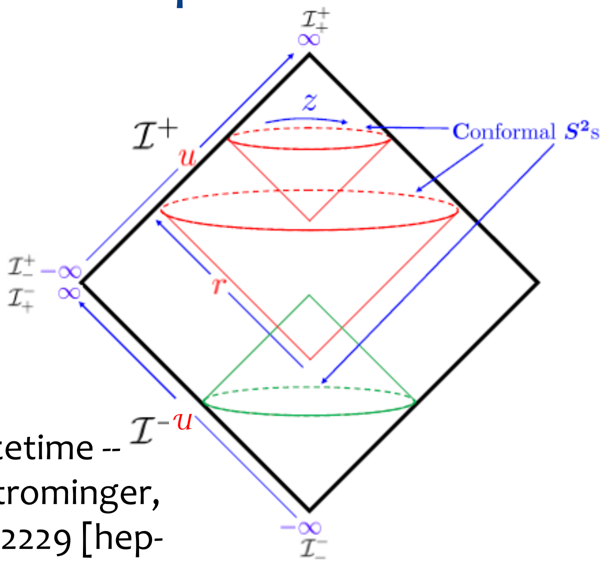
* The leading constraint equation is

$$2\partial_u F_{ur}^{(C\pm,d)} = e^2 J_u^{(\pm,d)}.$$

Matching conditions

- To study scattering, we need to relate the field strength near $\mathcal{I}^+$ to that near $\mathcal{I}^-$. This is done in a natural way such that Lorentz invariance is preserved.
- We get the matching conditions

$$F_{ur}^{(+,d)}\Big|_{\mathscr{I}^+_-} = -\,F_{ur}^{(-,d)}\Big|_{\mathscr{I}^-_+}$$



In 4D spacetime -- From A. Strominger, arXiv:1312.2229 [hep-th]

Conserved charge

- The matching condition immediately implies the following quantity is conserved:

$$Q_\varepsilon^\pm = \pm \frac{2}{e^2} \int_{\mathscr{I}^\pm_\mp} d^d x \, \varepsilon F_{ur}^{(\pm,d)} \qquad\qquad Q_\varepsilon^+ = Q_\varepsilon^-$$

- This precisely generates the large gauge transforms in QED parametrized by ε.
- When ε=1, this is the usual charge in QED defined via Gauss' law.

Soft and hard charges

* We decompose the charge into soft and hard parts:

$$Q_\varepsilon^\pm = Q_\varepsilon^{\pm S} + Q_\varepsilon^{\pm H}$$

  where

$$Q_\varepsilon^{\pm S} = \pm\frac{2}{e^2}\int_{\mathscr{I}^\pm_\mp} d^d x\, \varepsilon F_{ur}^{(R\pm,d)}$$

$$Q_\varepsilon^{\pm H} = -\int_{\mathscr{I}^\pm} du d^d x\, \varepsilon J_u^{(\pm,d)}$$

* The form of the soft charge is dimension dependent.

Radiative field

* In flat null coordinates, we can perform mode expansion to get

$$F_{ur}^{(R\pm)}(u,r,x) = \frac{e}{4(2\pi)^{d+1}r}\int_0^\infty d\omega \int d^d y\, \omega^{d-1}\left[\partial^a \mathcal{O}_a^{(\pm)}(\omega, x+y)e^{-\frac{i}{2}\omega u - \frac{i}{2}\omega r y^2} + \text{c.c.}\right]$$

* We assume the creation and annihilation operators admit a soft expansion

$$\partial^a \mathcal{O}_a^{(\pm)}(\omega, x+y) = \sum_{n=0}^{\infty} \omega^{n-1}\partial^a \mathcal{O}_a^{(\pm,n)}(x+y) \qquad \text{as} \qquad \omega \to 0^+.$$

Radiative field – Odd dimensions

* In odd dimensions, this is

$$F_{ur}^{(R\pm)}(u,r,x) = \frac{e}{8(2\pi)^{\frac{d}{2}}} \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \frac{2^{\nu_n - 2s} \csc(\pi\nu_n)}{\Gamma(s+1)\Gamma(1+s-\nu_n)} \times \left[\frac{i(iu)^{s-\nu_n}}{(ir)^{\frac{d}{2}+1+s}} (-\partial^2)^s \partial^a \mathcal{O}_a^{(\pm,n)}(x) + \text{c.c.} \right] + \frac{e}{2^{\frac{d}{2}+3}\pi^{d+1}} \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} (-1)^s \frac{\Gamma\left(\frac{d}{2}+\nu_n+s\right)}{2^{-\nu_n}\Gamma(s+1)} \times \left[\frac{i(iu)^s}{(ir)^{d+n+s}} \int d^d y \frac{\partial^a \mathcal{O}_a^{(\pm,n)}(y)}{[(x-y)^2]^{\frac{d}{2}+\nu_n+s}} + \text{c.c.} \right]$$

$$\nu_n = \frac{d}{2} - 1 + n \notin \mathbb{Z}$$

Radiative field – Even dimensions

- In even dimensions, this is

$$
\begin{aligned}
&F_{ur}^{(R\pm)}(u,r,x)\\
&= \frac{e}{2(2\pi)^{\frac{d}{2}+1}} \sum_{n=0}^{\infty} \sum_{s=0}^{\nu_n-1} \frac{(-1)^s \Gamma(\nu_n - s)}{2^{2s-\nu_n+1}\Gamma(s+1)} \left[\frac{i(iu)^{s-\nu_n}}{(ir)^{\frac{d}{2}+1+s}} (-\partial^2)^s \partial^a \mathcal{O}_a^{(\pm,n)}(x) + \text{c.c.} \right]\\
&\quad + \frac{e}{8\pi} \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^s 2^{\frac{d}{2}+\nu_n} \Gamma\left(\frac{d}{2}+s+\nu_n\right)}{\Gamma(s+1)} \left[\frac{i(iu)^s}{(ir)^{d+n+s}} \int d^d y \frac{\partial^a \mathcal{O}_a^{(\pm,n)}(y)}{[(x-y)^2]^{\frac{d}{2}+\nu_n+s}} + \text{c.c.} \right]\\
&\quad + \frac{e}{2(2\pi)^{\frac{d}{2}+1}} \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \frac{2^{-2s-\nu_n}(-1)^{\nu_n-1}}{\Gamma(s+1)\Gamma(s+\nu_n+1)}\\
&\qquad \times \left[\frac{i(iu)^s \left[\log\left(\frac{1}{2}\frac{\sqrt{iu}}{\sqrt{ir}} e^{\gamma_E}\right) - \frac{1}{2}\left(H_s + H_{s+\nu_n}\right)\right]}{(ir)^{d+n+s}} (-\partial^2)^{s+\nu_n} \partial^a \mathcal{O}_a^{(\pm,n)}(x) + \text{c.c.} \right].
\end{aligned}
$$

Extracting the coefficient

- In odd dimensions the coefficient of $1/r^d$ in F_{ur} is

$$F_{ur}^{(R\pm,d)}(u,x)=\frac{e(-1)^{\frac{d-1}{2}}\Gamma\left(d-1\right)}{16\pi^{d+1}}\int d^d y\frac{\partial^a\mathcal{O}_a^{(\pm,0)}(y)+\partial^a\mathcal{O}_a^{(\pm,0)\dagger}(y)}{\left[(x-y)^2\right]^{d-1}}$$

- In even dimensions the coefficient of $1/r^d$ in F_{ur} is

$$F_{ur}^{(R\pm,d)}=-\frac{e[\Theta(u)-\Theta(r)]}{8(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)}\left(-\partial^2\right)^{\frac{d}{2}-1}\left[\partial^a\mathcal{O}_a^{(\pm,0)}(x)+\partial^a\mathcal{O}_a^{(\pm,0)\dagger}(x)\right]$$
$$+\left[\partial^a\mathcal{O}_a^{(\pm,0)}(x)-\partial^a\mathcal{O}_a^{(\pm,0)\dagger}(x)\right]\left[\cdots\right]+O\left(u^{-1}\right)$$

Removing log divergences

* To get rid of log divergences in even dimensions so that the charge is well-defined, we assume

$$\partial^a \mathcal{O}_a^{(\pm,0)}(x) = \partial^a \mathcal{O}_a^{(\pm,0)\dagger}(x)$$

* This means the strict zero energy limit of both the creation and annihilation operator just produce a new vacuum state.

Properties of the coefficient

* In odd dimensions, there is nonzero flux through i^+.

$$F_{ur}^{(R\pm,d)}(u,x) = \frac{e(-1)^{\frac{d-1}{2}}\Gamma\left(d-1\right)}{16\pi^{d+1}}\int d^d y \frac{\partial^a \mathcal{O}_a^{(\pm,0)}(y) + \partial^a \mathcal{O}_a^{(\pm,0)\dagger}(y)}{\left[(x-y)^2\right]^{d-1}}$$

* In even dimensions the coefficient of $1/r^d$ in F_{ur} is

$$F_{ur}^{(R\pm,d)} = -\frac{e[\Theta(u)-\Theta(r)]}{8(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)}\left(-\partial^2\right)^{\frac{d}{2}-1}\left[\partial^a \mathcal{O}_a^{(\pm,0)}(x) + \partial^a \mathcal{O}_a^{(\pm,0)\dagger}(x)\right]$$
$$+ \left[\partial^a \mathcal{O}_a^{(\pm,0)}(x) - \partial^a \mathcal{O}_a^{(\pm,0)\dagger}(x)\right][\cdots] + O\left(u^{-1}\right)$$

Soft charge

* With the assumption, the soft charge is

$$Q_\varepsilon^{\pm S} = \begin{cases} \frac{1}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)e} \int d^d x\, \varepsilon(x)(-\partial^2)^{\frac{d}{2}-1}\partial^a \mathcal{O}_a^{(\pm,0)}(x), & d \in 2\mathbb{Z}, \\ \frac{(-1)^{\frac{d-1}{2}}\Gamma(d-1)}{4\pi^{d+1}e} \int d^d x\, \varepsilon(x) \int d^d y \frac{\partial^a \mathcal{O}_a^{(\pm,0)}(y)}{\left[(x-y)^2\right]^{d-1}}, & d \in 2\mathbb{Z}+1 \end{cases}$$

* Choose $\varepsilon(z) = (-\partial^2)\log\left[(x-z)^2\right]$ to get

$$Q_\varepsilon^{\pm S} = -\frac{1}{2e}\left(\partial^a \mathcal{O}_a^{(\pm,0)}(x) + \partial^a \mathcal{O}_a^{(\pm,0)}(x)^\dagger\right) = -\frac{1}{e}\partial^a \mathcal{O}_a^{(\pm,0)}(x)$$

Deriving the Ward identity

* The charge Q_ε is conserved in the scattering process due to the matching condition, so it must commute with the S-matrix. Hence, the Ward identity corresponding to the asymptotic symmetry is simply

$$\langle \text{out}|(Q_\varepsilon \mathcal{S} - \mathcal{S} Q_\varepsilon)|\text{in}\rangle = 0.$$

Obtaining the Ward identity

- Decomposing the charge into soft and hard pieces, and acting on the in and out states, we get

$$\langle \text{out}| \left[\partial^a \mathcal{O}_a^{(+,0)}(x)\mathcal{S} - \mathcal{S}\partial^a \mathcal{O}_a^{(-,0)\dagger}(x)\right] |\text{in}\rangle = e\sum_{i=1}^{n} \eta_i Q_i \partial^2 \log\left[(x-x_i)^2\right] \langle \text{out}|\mathcal{S}|\text{in}\rangle$$

- We claim this is the soft theorem in flat null coordinates in disguise.

Connecting to soft photon theorem

* Recall the leading soft photon theorem is

$$\lim_{p_\gamma \to 0} \mathcal{A}^{\text{out}}_{n+1}(\vec{p}_\gamma, \varepsilon; p_1, \ldots, p_n) = e \sum_{i=1}^{n} \eta_i Q_i \frac{p_i \cdot \varepsilon(p_\gamma)}{p_i \cdot p_\gamma} \mathcal{A}_n(p_1, \ldots, p_n)$$

* Using the LSZ reduction formula, inserting an outgoing soft photon corresponds to the operator insertion

$$\lim_{\omega \to 0} \left[\omega \mathcal{O}^{(+)}_a(\omega, x) - \omega \mathcal{O}^{(-)}_a(\omega, x)\right] = \mathcal{O}^{(+,0)}_a(x) - \mathcal{O}^{(-,0)}_a(x)$$

Connecting to soft photon theorem

* Switching to the flat null coordinates, the soft photon theorem becomes

$$\langle \text{out}| \left[\mathcal{O}_a^{(+,0)}(x)\mathcal{S} - \mathcal{S}\mathcal{O}_a^{(-,0)}(x) \right] |\text{in}\rangle = e \sum_{i=1}^{n} \eta_i Q_i \partial_a \log\left[(x - x_i)^2 \right] \langle \text{out}|\mathcal{S}|\text{in}\rangle$$

* Taking the derivative of both sides gives us precisely the Ward identity derived two slides ago!

Summary

* Performed an asymptotic expansion of F_{ur} in both odd and even dimensions near $\mathcal{I}^{\pm}$
* Extracted the $1/r^d$ coefficient in F_{ur} and wrote down the corresponding soft and hard charges
* Used the matching condition to write down the Ward identity
* Rewrote the leading soft theorem in terms of flat null coordinates and showed it matched the Ward identity

Ongoing research

* We can readily generalize this to the subleading soft photon theorem in arbitrary dimensions, as well as nonabelian gauge theory (see 1903.03607).
* We want to generalize this analysis to gravity as well.
* Memory effects in odd dimensions is being explored.
* We want to better understand the assumption

$$\partial^a \mathcal{O}_a^{(\pm,0)}(x) = \partial^a \mathcal{O}_a^{(\pm,0)\dagger}(x)$$

Thank you for listening!