# Dr. Cardy or: How I Learned to Stop Worrying and Love Effective Actions

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Based on *arXiv:1910.10151* $\oplus$ WIP $\oplus$ $\cdots$ with Chi-Ming Chang, Ying-Hsuan Lin and Yifan Wang.

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The following is (mostly) concerned with the "old" 4d/6d SUSY Cardy formula ("real fugacities"), first conjectured by [*Di Pietro-Komargodski*]. The techniques outlined here are more general however, and can be applied to the "new limit" [*Choi-Kim$^2$-Nahmgoong; Cabo-Bizet-Cassini-Martelli-Murthy;* $\cdots$], which seems to be connected to black hole microstate counting. Time-permitting, I will say a few words about this at the end.

# Game plan:

1. Motivation: 2d Cardy formula
2. Higher-dimensional Cardy formulae
   - Preliminaries
   - SUSY Cardy formulae: A conjecture
3. Proof:
   - Effective actions
   - 5d Chern-Simons invariants
   - Chern-Simons couplings from EA
   - Examples
4. Global anomalies from EA
5. Some corollaries
6. Summary and outlook

# Motivation: 2d Cardy formula

- Torus partition function:

$$\mathcal{Z}_{S^1_\beta \times S^1}(\tau) \;=\; \mathrm{Tr}_{\mathcal{H}_{S^1}}\left(q^{L_0-\frac{c}{24}}\,\bar{q}^{\bar{L}_0-\frac{c}{24}}\right), \qquad \tau \;=\; \frac{\mathrm{i}\beta}{2\pi}$$

- Cardy limit $\beta \to 0$ [Cardy]:

$$\log \mathcal{Z}_{S^1_\beta \times S^1} \;=\; \frac{\pi^2 \mathbf{c}}{3\beta} + \mathcal{O}(\beta^0, \log\beta)$$

- **Key Ingredient:**
  - ⋆ Modular invariance: $Z(\tau) = Z(-1/\tau)$
- **Consequences/Features:**
  - (i) Operator spectrum $\rho(\Delta)$ in the high-energy asymptotic region
  - (ii) Universality (only dependent on **c**)
  - (iii) Under the AdS/CFT: maps to the universality of the Bekenstein-Hawking entropy of BTZ black holes [Strominger-Vafa]
- **Recent resurgence:**
  - (iv) Recent refinements using Tauberian theorems [Mukhametzhanov-Zhiboedov; ⋯]
  - (v) Modular bootstrap
  - ...

# Preliminaries: Higher-dimensional Cardy formulae?

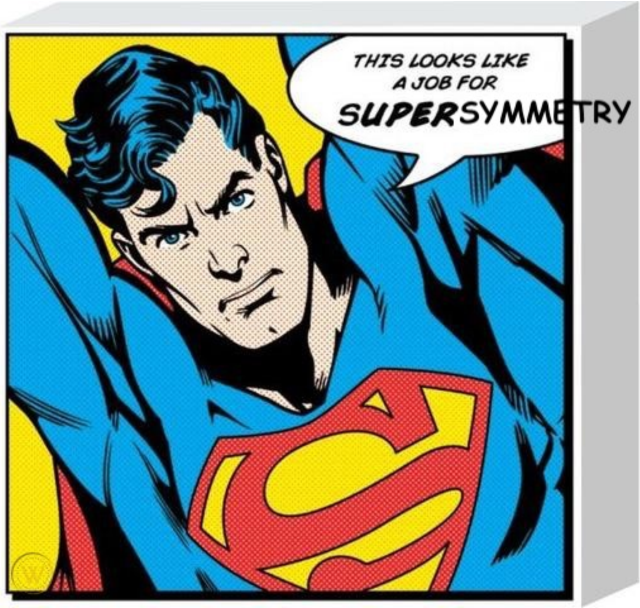
**Some considerations:**

I. **Spacetime?** Natural choices: $\mathbb{T}^d$, $\mathbb{T}^2 \times S^{d-2}$, $S^1 \times S^{d-1}$, *etc*

II. **Modularity?** Unknown in general (some progress for $S^1 \times S^3$ in [*Dedushenko-MF*])

III. **Universality?** Dependence on spacetime/chemical potentials seems complicated/non-universal

Does high-temperature universality exist in $d > 2$?




[1]Idea blatantly stolen from Christoph

# A conjecture in 4d

- $\mathcal{N}=1$ SUSY index ($Q$ and its conjugate $Q^\dagger$ generates an $\mathfrak{su}(1|1)$):

$$\mathcal{Z}_{S^1_\beta \times S^3} = \mathrm{Tr}_{\mathcal{H}} \left[ (-1)^F e^{-L\{Q,Q^\dagger\}} e^{-\beta \sum_{i=1}^2 \omega_i (j_i + R)} \right]$$

- In the Cardy limit $\beta \to 0$, this SUSY partition function has the expansion [*Di Pietro-Komargodski*]:

$$\log \mathcal{Z}_{S^1_\beta \times S^3} = \frac{\pi^2}{6\beta} \frac{\omega_1 + \omega_2}{\omega_1 \omega_2} \kappa + \mathcal{O}(\beta^0, \log \beta)$$

- $\kappa$ related to the anomaly coefficient $k$ for the mixed gravitational-R-symmetry by

$$\kappa = -k$$

- The anomaly coefficient $k$ appears in the anomaly polynomial 6-form as

$$I_6 \ni \frac{k}{48(2\pi)^3} F_R \wedge \mathrm{tr}\,(R \wedge R)$$

- By SUSY, $\kappa$ and $k$ are in turn related to the 4d conformal anomalies as

$$\kappa = -k = 16(c-a)$$

# A conjecture in 6d

- $\mathcal{N} = (1,0)$ 6d index:

$$\mathcal{Z}_{S^1_\beta \times S^5} = \mathrm{Tr}_{\mathcal{H}} \left[ (-1)^F e^{-L\{Q,Q^\dagger\} - \beta \sum_I \mu_{\mathrm{f}}^I H_{\mathrm{f}}^I - \beta \sum_{i=1}^3 \omega_i (j_i + R)} \right]$$

- [*Di Pietro-Komargodski*] conjectured a Cardy formula for the Cardy limit $\beta \to 0$ (based on free field examples):

$$\log \mathcal{Z}_{S^1_\beta \times S^5} = -\frac{\pi}{\omega_1\omega_2\omega_3} \left[ \frac{\kappa_1}{360} \left(\frac{2\pi}{\beta}\right)^3 + \frac{(\omega_1^2+\omega_2^2+\omega_3^2)(\kappa_2 - 3\kappa_3/2)}{72} \left(\frac{2\pi}{\beta}\right) + \frac{(\omega_1+\omega_2+\omega_3)^2 \kappa_3}{48} \left(\frac{2\pi}{\beta}\right) + \frac{\mu_{\mathrm{f}}^2 \kappa_{\mathrm{f}}^{G_{\mathrm{f}}}}{24} \left(\frac{2\pi}{\beta}\right) \right] + \mathcal{O}(\beta^0, \log\beta)$$

- $\kappa_i$ fixed by the perturbative anomalies

$$\kappa_1 = -40\gamma - 10\delta\,, \quad \kappa_2 - \frac{3}{2}\kappa_3 = 16\gamma - 2\delta\,, \quad \kappa_3 = -2\beta\,, \quad \kappa_{\mathrm{f}}^{G_{\mathrm{f}}} = -48\mu^{G_{\mathrm{f}}}\,.$$

- Anomaly coefficients in anomaly polynomial 8-form:

$$I_8 = \frac{1}{4!} \left[ \alpha c_2(SU(2)_R)^2 + \beta c_2 p_1 + \gamma p_1^2 + \delta p_2 \right] + \mu^{G_{\mathrm{f}}} p_1 c_2(G_{\mathrm{f}})$$

# Status of the Cardy formulae

**4d:**

- [*Di Pietro-Komargodski*] derive the 4d formula based on Lagrangian theories (*i.e.* there exists a point in the space of continuous couplings where the theory becomes free)
  - ⊕ checks for Lagrangian theories from localization [*Di Pietro-Komargodski; Ardehali; Di Pietro-Honda*]
  - ⊕ Schur index [*Ardehali; Buican-Nishinaka*]
- However, ∃ non-Lagrangian theories:
  - Formula holds based on non-Lagrangian examples [*Buican-Nishinaka*]
  - Modularity properties from "chiral algebra/4d $\mathcal{N} = 2$" correspondence [*Beem-Rastelli*]

**6d:**

- [*Di Pietro-Komargodski*] conjectured formula based on free multiplets

# Effective actions

6d/4d SCFT
$(A_1, A_2) = AD_{N_f=1}(SU(2))$ AD theory

↓ on vacuum moduli space
Turn on VEV: $\langle \Phi \rangle \sim m_{moduli}$

6d/4d *Wilsonian* effective action
Cutoff $\Lambda$: $m_{moduli} < \Lambda$, $E < \Lambda$
BPS quiver: ◯→◯

→ $\int$ out massive fields

6d/4d effective action:
$E < m_{moduli}$
free ml $U(1)$ vm

↓ reduce on $S^1$
$\int$ out massive fields

5d/3d *Wilsonian* effective action
Cutoff $\Lambda$: $m_{moduli} < \Lambda < \beta^{-1}$, $E < \Lambda$

⇣ reduce on $S^1$
$\int$ out all fields

→ $\int$ out ml fields

5d/3d effective action
$\kappa_1 A \wedge \mathrm{d}A \wedge \mathrm{d}A + \cdots$

# 3d Chern-Simons Effective Action:

- 4d Metric:

$$\mathrm{d}s_4^2 \;=\; \left(\mathrm{d}\tau + \frac{\beta}{2\pi} A_i \mathrm{d}x^i\right)^2 + h_{ij}\mathrm{d}x^i \mathrm{d}x^j$$

- 3d Chern-Simons Effective Action

$$\mathrm{i}W = -\log \mathcal{Z} = \frac{\mathrm{i}\kappa}{24\pi}\left(-\int V_R \wedge \mathrm{d}A + \text{ SUSY completion}\right) + \mathcal{O}(\beta^0, \log\beta)\,,$$

where the graviphoton $A$ is of order $\mathcal{O}(\beta^{-1})$.

# 5d Chern-Simons Effective Action:

- 6d Metric:

$$\mathrm{d}s_6^2 = \left(\mathrm{d}\tau + \frac{\beta}{2\pi} A_i \mathrm{d}x^i\right)^2 + h_{ij}\mathrm{d}x^i\mathrm{d}x^j$$

- 5d Chern-Simons Effective Action:

$$\mathrm{i}W = -\log \mathcal{Z} = \frac{\mathrm{i}}{8\pi^2}\left(\frac{\kappa_1}{360} I_1 + \frac{\kappa_2 - \frac{3}{2}\kappa_2}{144} I_2 - \frac{\kappa_3}{24} I_3 - \frac{\kappa_{\mathrm{f}}^{G_{\mathrm{f}}}}{24} I_4^{G_{\mathrm{f}}}\right) + \mathcal{O}(\beta^0, \log\beta)$$

- where the "counterterms" (classified in [Chang-MF-Lin-Wang]) are

$$I_1 \equiv \int A \wedge \mathrm{d}A \wedge \mathrm{d}A + \text{ SUSY completion},$$

$$I_2 \equiv \int A \wedge \mathrm{tr}\,(R \wedge R) + \text{ SUSY completion},$$

$$I_3 \equiv \int A \wedge \mathrm{Tr}\,(F_R \wedge F_R) + \text{ SUSY completion},$$

$$I_4^{G_{\mathrm{f}}} \equiv \int A \wedge \mathrm{Tr}\,(F_{G_{\mathrm{f}}} \wedge F_{G_{\mathrm{f}}}) + \text{ SUSY completion}.$$

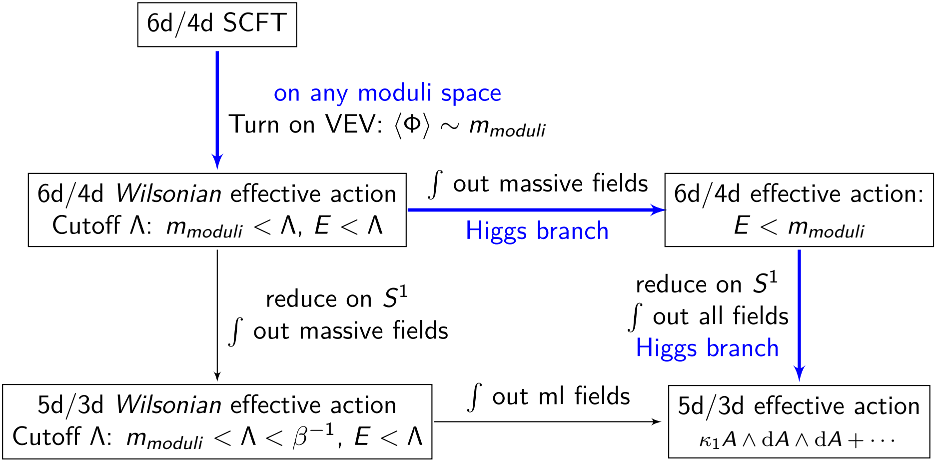
Here, $A$ is the $U(1)_{\mathrm{KK}}$ graviphoton (which in the $\beta \to 0$ limit scales as $\beta^{-1}$), $R$ denotes the Riemann curvature 2-form of the 5d background metric $h_{ij}$.

# Contributions from massive BPS particles:

- 4d $\mathcal{N} = 2$:
  - HB: BPS strings
  - CB: BPS particles
  - Mixed: BPS particles, strings and domain-walls

- 6d $\mathcal{N} = (1,0)$:
  - HB: BPS codimension-two branes
  - TB: BPS strings
  - Mixed: BPS strings and codimension-two branes

# Effective actions (again)



# CS invariants: SC multiplets

- **5d Standard Weyl multiplet:**

$$\mathcal{SW} \; = \; \left(g_{\mu\nu}\;,\; D\;,\; V^{ij}_\mu\;,\; v_{\mu\nu}\;,\; b_\mu\;,\; \psi^i_\mu\;,\; \chi^i\right)$$

SUSY-conditions:

$$\begin{aligned}\delta\psi^i_\mu \; &= D_\mu\varepsilon^i + \frac{1}{2}v^{\nu\rho}\gamma_{\mu\nu\rho}\varepsilon^i - \gamma_\mu\eta^i\,,\\ \delta\chi^i \; &= \varepsilon^i D - 2\gamma^\rho\gamma^{\mu\nu}\varepsilon^i\nabla_\mu v_{\nu\rho} + \gamma^{\mu\nu}F_{\mu\nu}{}^i{}_j(V)\varepsilon - 2\gamma^\mu\varepsilon^i\epsilon_{\mu\nu\rho\sigma\lambda}v^{\nu\rho}v^{\sigma\lambda}\\ &\quad + 4\gamma^{\mu\nu}v_{\mu\nu}\eta^i\,.\end{aligned}$$

- **5d Vector multiplet coupled to $\mathcal{SW}$:**

$$\mathcal{V} \; = \; \left(W_\mu\;,\; M\;,\; \Omega^i_\alpha\;,\; Y^{ij}\right)$$

SUSY-conditions:

$$\delta\Omega^i \; = \; -\frac{1}{4}\gamma^{\mu\nu}F_{\mu\nu}(W)\varepsilon^i - \frac{1}{2}\not{D}M\varepsilon^i + Y^i{}_j\varepsilon^j - M\eta^j$$

- **5d Linear multiplet coupled to $\mathcal{SW}$ (compensator):**

$$\mathcal{L} \; = \; \left(L_{ij}\;,\; \varphi^i_\alpha\;,\; E^\mu\;,\; N\right)$$

SUSY-conditions:

$$\delta\varphi^i \; = \; -\not{D}L^i{}_j\varepsilon^j + \frac{1}{2}\gamma^\mu\varepsilon^i E_\mu + \frac{1}{2}\varepsilon^i N + 2\gamma^{\mu\nu}v_{\mu\nu}\varepsilon^j L^i{}_j - 6L^{ij}\eta_j$$

# CS invariants: Poincaré supergravity

Gauge-fixing $\mathcal{SW}$ together with compensators $\mathcal{V}$ and $\mathcal{L}$: Different choices:

$$\mathcal{P}_k \equiv \frac{\left(g_{\mu\nu}, D, V_\mu^{ij}, v_{\mu\nu}, b_\mu, \psi_\mu^i, \chi^i\right) \oplus \left(W_\mu, M, \Omega_\alpha^i, Y^{ij}\right) \oplus \left(L_{ij}, \varphi_\alpha^i, E^\mu, N\right)}{(\text{Gauge fixing})_k}$$

f **"Standard/flavor gauge":** scalar = cst ≡ flavor mass $m_\mathrm{f}$

$$\underbrace{\hat{L}^i{}_j = \frac{\mathrm{i}}{2}\hat{L}(\sigma_3)^i{}_j}_{\mathfrak{su}(2)_R \to \mathfrak{u}(1)_R}, \quad \underbrace{\hat{M} = m_\mathrm{f}}_{Dilatations}, \quad \underbrace{b_\nu = 0}_{spec.\ conf.}, \quad \underbrace{\hat{\Omega}^i = 0}_{\mathcal{S}\ SUSY}$$

$$\mathcal{P}_\mathrm{f} = \left(g_{\mu\nu}, D, V_\mu^{12}, v_{\mu\nu}, \hat{W}_\mu, \hat{Y}^{12}, \hat{E}^\mu, \hat{L}, \hat{N}, \psi_{\mu\alpha}^i, \chi_\alpha^i, \hat{\varphi}_\alpha^i\right).$$

KK **"KK gauge":** gauge field = KK/graviphoton = $m_\mathrm{KK}\mathcal{Y}$, with $m_\mathrm{KK} \sim$ warping factor.[2]

$$\underbrace{\hat{L}^i{}_j = \frac{\mathrm{i}}{2}\hat{L}(\sigma_3)^i{}_j}_{\mathfrak{su}(2)_R \to \mathfrak{u}(1)_R}, \quad \underbrace{\hat{L} = 1}_{Dilatations}, \quad \underbrace{b_\nu = 0}_{spec.\ conf}, \quad \underbrace{\hat{\varphi}^i = 0}_{\mathcal{S}\ SUSY}.$$

$$\mathcal{P}_\mathrm{KK} = \left(g_{\mu\nu}, D, V_\mu^{12}, v_{\mu\nu}, \hat{M}, \hat{W}_\mu, \hat{Y}^{12}, \hat{E}^\mu, \hat{N}, \psi_{\mu\alpha}^i, \chi_\alpha^i, \hat{\Omega}_\alpha^i\right).$$

# Interlude: Rigid SUSY on $\mathcal{M}_5$

Rigid SUSY for *Standard Weyl* [*Alday,Benetti,MF,Richmond,Sparks*] and *gauge-fixed Poincaré* [*unpublished*]:

$\Leftrightarrow$ We obtain that $(M_5, g)$ is equipped with a conformal Killing vector generating a *transversally holomorphic foliation* (THF). The transverse metric $g_4$ is an arbitrary Hermitian metric with respect to the transverse complex structure + explicit equations for background fields.

**Some examples:**

- Product metric $M_5 = \mathbb{R} \times M_4$ or $M_5 = S^1 \times M_4$, where $M_4$ is Hermitian.
- Circle bundle over a product of Riemann surfaces $S^1 \hookrightarrow \Sigma_1 \times \Sigma_2$.
  - If we only fibre over $\Sigma_1$, this leads to direct product $M_3 \times \Sigma_2$ solutions, where $M_3$ is a Seifert fibred three-manifold.
- Sasakian
- Squashed five-spheres
- $S^1 \times S^4$ only seems to exist in conformal supergravity.
- Twisted indices
- $\cdots$

# Geometric invariants & their evaluation on squashed $S^5$:

We now fix (for simplicity) the space to be $S^1 \times S^5_{\rm sq}$, with the 6d metric:

$$\mathrm{d}s^2_{S^1\times S^5} \;=\; r_5^2 \sum_{i=1}^{3} \left[\mathrm{d}y_i^2 + y_i^2 \left(\mathrm{d}\phi_i + \frac{\mathrm{i}a_i}{r_5}\mathrm{d}\tau\right)^2\right] + \mathrm{d}\tau^2\,,$$

The corresponding 5d metric is then:

$$\mathrm{d}s_5^2 \;=\; \sum_{i=1}^{3}\left(\mathrm{d}y_i^2 + y_i^2\mathrm{d}\phi_i^2\right) + \tilde{\kappa}^{-2}\mathcal{Y}^2\,,$$

$$\mathcal{Y} \;=\; \tilde{\kappa}^2 \sum_{i=1}^{3} a_i y_i^2 \mathrm{d}\phi_i\,, \qquad \tilde{\kappa}^{-2} \;=\; 1 - \sum_{j=1}^{3} y_j^2 a_j^2\,,$$

where now

$$A \;=\; m_{\rm KK}\, r_5 \mathcal{Y}\,, \qquad m_{\rm KK} \;=\; \frac{2\pi\mathrm{i}}{\beta}\,.$$

and we now proceed to the evaluation of $I_1, \ldots I_4$ in this background.

# Geometric invariants & their evaluation on squashed $S^5$:

**EA:** $$\mathrm{i}W = \frac{\mathrm{i}}{8\pi^2}\left(\frac{\kappa_1}{360}I_1 + \frac{\kappa_2 - \frac{3}{2}\kappa_2}{144}I_2 - \frac{\kappa_3}{24}I_3 - \frac{\kappa_{\mathrm{f}}^{G_{\mathrm{f}}}}{24}I_4^{G_{\mathrm{f}}}\right) + \mathcal{O}(\beta^0, \log\beta)$$

$$\begin{aligned}
I_1, I_4^{G_{\mathrm{f}}} &\equiv \int A\wedge \mathrm{d}A \wedge \mathrm{d}A + \text{ SUSY completion} \\
&\equiv \text{SUSY Euclidean vm's, } \{\mathcal{V}_I\}_I\text{, action}^3 \text{ coupled to } \mathcal{SW} \\
&= S_{\mathrm{vm}}\left(\mathcal{V}_I, \mathcal{V}_J, \mathcal{V}_K\right) \\
&= \int_{\mathcal{M}_5} c_{IJK}\Bigg[\frac{1}{2}W^I\wedge F^J(W)\wedge F^K(W) - \frac{3}{2}M^I F^J(W)\wedge *F^K(W) \\
&\qquad + \frac{3}{2}M^I\mathrm{d}M^J\wedge *\mathrm{d}M^K - 3M^I M^J\left(2F^K(W) + M^K v\right)\wedge * v \\
&\qquad + M^I\left(3(Y^J)_{ij}(Y^K)^{ij} + \frac{1}{4}M^J M^K\left[\frac{R}{2} - D\right]\right)\mathrm{vol}_5\Bigg]
\end{aligned}$$

[3] [Bergshoeff-de Wit-Halbersma-Cucu-Derix-Van Proeyen; Fujita-Ohashi]

# Geometric invariants & their evaluation on squashed $S^5$:

**EA:** $$\mathrm{i}W = \frac{\mathrm{i}}{8\pi^2}\left(\frac{\kappa_1}{360}\mathbf{I_1} + \frac{\kappa_2 - \frac{3}{2}\kappa_2}{144} I_2 - \frac{\kappa_3}{24} I_3 - \frac{\kappa_{\mathrm{f}}^{G_{\mathrm{f}}}}{24} I_4^{G_{\mathrm{f}}}\right) + \mathcal{O}(\beta^0, \log\beta)$$

$$\begin{aligned}
I_1 &\equiv \int A_{\mathrm{KK}} \wedge \mathrm{d}A_{\mathrm{KK}} \wedge \mathrm{d}A_{\mathrm{KK}} + \text{ SUSY completion} \\
&\equiv \text{SUSY Euclidean vm, } \mathcal{V}_{\mathrm{KK}}\text{, action coupled to } \mathcal{SW} \\
&= S_{\mathrm{vm}}\left(\mathcal{V}_{\mathrm{KK}}, \mathcal{V}_{\mathrm{KK}}, \mathcal{V}_{\mathrm{KK}}\right) \\
&= \frac{m_{\mathrm{KK}}^3}{2}\int_{\mathcal{M}_5} \eta \wedge \mathrm{d}\eta \wedge \mathrm{d}\eta + \int_{\mathcal{M}_5} \mathrm{d} * (\cdots) \\
&= \frac{m_{\mathrm{KK}}^3}{2}\frac{(2\pi)^3}{\omega_1\omega_2\omega_3}
\end{aligned}$$

# Geometric invariants & their evaluation on squashed $S^5$:

**EA:** $$\mathrm{i}W = \frac{\mathrm{i}}{8\pi^2}\left(\frac{\kappa_1}{360}\mathbf{I_1} + \frac{\kappa_2 - \frac{3}{2}\kappa_2}{144} I_2 - \frac{\kappa_3}{24} I_3 - \frac{\kappa_{\mathrm{f}}^{G_{\mathrm{f}}}}{24}\mathbf{I_4^{G_f}}\right) + \mathcal{O}(\beta^0, \log\beta)$$

$$\begin{aligned}
I_4^{G_{\mathrm{f}}} &\equiv \int A_{\mathrm{f}} \wedge \mathrm{d}A_{\mathrm{f}} \wedge \mathrm{d}A_{\mathrm{KK}} + \text{ SUSY completion} \\
&\equiv \text{SUSY Euclidean vms, } \left\{\mathcal{V}_{\mathrm{KK}}, \mathcal{V}_{\mathrm{f}}^{\ell}\right\}\text{, action coupled to } \mathcal{SW} \\
&= S_{\mathrm{vm}}\left(\mathcal{V}_{\mathrm{KK}}, \mathcal{V}_{\mathrm{f}}, \mathcal{V}_{\mathrm{f}}\right) \\
&= \frac{m_{\mathrm{KK}}\mu_{\mathrm{f}}^2}{2}\int_{\mathcal{M}_5} \eta \wedge \mathrm{d}\eta \wedge \mathrm{d}\eta + \int_{\mathcal{M}_5} \mathrm{d} * (\cdots) \\
&= \frac{m_{\mathrm{KK}}}{2}\frac{(2\pi)^3}{\omega_1\omega_2\omega_3}\mu_{\mathrm{f}}^2
\end{aligned}$$

# Geometric invariants & their evaluation on squashed $S^5$:

**EA:** $$\mathrm{i}W = \frac{\mathrm{i}}{8\pi^2}\left(\frac{\kappa_1}{360}\mathbf{I_1} + \frac{\kappa_2 - \frac{3}{2}\kappa_2}{144} I_2 - \frac{\kappa_3}{24} I_3 - \frac{\kappa_{\mathrm{f}}^{G_{\mathrm{f}}}}{24}\mathbf{I_4^{G_f}}\right) + \mathcal{O}(\beta^0, \log\beta)$$

Remains $I_2 = \int A \wedge \mathrm{tr}\,(R \wedge R) +$ SUSY and $I_3 = \int A \wedge \mathrm{Tr}\,(F_R \wedge F_R) +$ SUSY.

- Both higher derivative terms
- there are 3 higher-derivative terms: $R^2$, $\mathrm{Ric}^2$ and $\mathrm{Riemann}^2$
- $\exists$ SUSY completion [*Hanaki-Ohashi-Tachikawa; Ozkan-Pang; Butter-Kuzenko-Novak-Tartaglino-Mazzucchelli*]
- field redefinitions $\implies$ 1 linear combination trivial
- 2 independent ones

  FWW: superconformal, $\mathcal{V} \oplus \mathcal{SW}$ [*Hanaki-Ohashi-Tachikawa*]

  FRR: Poincaré, $\mathcal{P}_{\mathrm{KK}} = \mathcal{SW} \oplus \mathcal{V} \oplus \mathcal{L}$/KK gauge [*Ozkan-Pang*]
- Wick rotate
- evaluate
- Then: $I_2 \equiv -4\left[\mathrm{FWW} - \frac{1}{6}\mathrm{FRR}\right], \qquad I_3 \equiv -\frac{1}{2}\mathrm{FRR}$

# Geometric invariants & their evaluation on squashed $S^5$:

**EA:** $$\mathrm{i}W = \frac{\mathrm{i}}{8\pi^2}\left(\frac{\kappa_1}{360}\mathbf{I_1} + \frac{\kappa_2 - \frac{3}{2}\kappa_2}{144} I_2 - \frac{\kappa_3}{24}\mathbf{I_3} - \frac{\kappa_{\mathrm{f}}^{G_{\mathrm{f}}}}{24}\mathbf{I_4^{G_f}}\right) + \mathcal{O}(\beta^0, \log\beta)$$

$$\begin{aligned} I_3 &\equiv \int_{\mathcal{M}_5} A \wedge \mathrm{tr}\,(F_R \wedge F_R) + \text{ SUSY completion} \\ &\equiv -\frac{1}{2}\Big(\text{SUSY Euclidean FRR action in terms of } \mathcal{P}_{\mathrm{KK}}\Big) \\ &= \text{long long expression – see paper} \\ &= \frac{(2\pi)^3}{2}\frac{(\omega_1+\omega_2+\omega_3)^2}{\omega_1\omega_2\omega_3} m_{\mathrm{KK}} \end{aligned}$$

# Geometric invariants & their evaluation on squashed $S^5$:

**EA:** $$\mathrm{i}W = \frac{\mathrm{i}}{8\pi^2}\left(\frac{\kappa_1}{360}\mathbf{I_1} + \frac{\kappa_2 - \frac{3}{2}\kappa_2}{144}\mathbf{I_2} - \frac{\kappa_3}{24}\mathbf{I_3} - \frac{\kappa_{\mathrm{f}}^{G_{\mathrm{f}}}}{24}\mathbf{I_4^{G_f}}\right) + \mathcal{O}(\beta^0, \log\beta)$$

$$\begin{aligned}
I_2 &\equiv \int_{\mathcal{M}_5} A \wedge \mathrm{tr}\,(R \wedge R) + \text{ SUSY completion} \\
&\equiv 4\Big(\text{SUSY Euclidean FWW action in terms of } \mathcal{SW} \oplus \mathcal{V}_{\mathrm{KK}}\Big) \\
&\quad + \frac{2}{3}\Big(\text{SUSY Euclidean FRR action in terms of } \mathcal{P}_{\mathrm{KK}}\Big) \\
&= -\frac{2}{3}\frac{(\omega_1+\omega_2+\omega_3)^2}{\omega_1\omega_2\omega_2}(2\pi)^3 m_{\mathrm{KK}}
\end{aligned}$$

$$\begin{aligned}
&+4\int_{\mathcal{M}_5} c_I\sqrt{g}\Big[-\tfrac{1}{12}\epsilon^{\mu\nu\rho\sigma\lambda}(W^I)_\mu\Big(\tfrac{3}{4}C_{\nu\rho\tau\delta}C_{\sigma\lambda}{}^{\tau\delta} - F(V)_{\nu\rho}{}^{ij}F(V)_{\sigma\lambda\, ij}\Big) \\
&M^I\Big(\tfrac{1}{8}C^2 - \big(v^2\big)^2 + \tfrac{1}{12}D^2\Big) + \Big(\tfrac{D}{6} - \tfrac{4}{9}v^2\Big)v_{\mu\nu}F^I(W)^{\mu\nu} + C_{\mu\nu\rho\sigma}v^{\mu\nu}\Big(\tfrac{1}{3}M^I v^{\rho\sigma} + \tfrac{1}{2}F^I(W)^{\rho\sigma}\Big) \\
&-\tfrac{4}{3}(Y^I)_{ij}F(V)_{\mu\nu}{}^{ij}v^{\mu\nu} - \epsilon_{\mu\nu\rho\sigma\lambda}F^I(W)^{\mu\nu}\Big(\tfrac{2}{3}v^{\rho\tau}\nabla_\tau v^{\sigma\lambda} + v^\rho{}_\tau\nabla^\sigma v^{\lambda\tau}\Big) \\
&-\tfrac{1}{3}M^I\Big(F(V)_{\mu\nu}{}^{ij}F(V)^{\mu\nu}{}_{ij} + 4\nabla_\nu v_{\mu\rho}\nabla^\mu v^{\nu\rho} - 8v_{\mu\nu}\nabla^\nu\nabla_\rho v^{\mu\rho}\Big) + \tfrac{1}{9}M^I\Big(16R^{\nu\rho}v_{\mu\nu}v^\mu{}_\rho + 12\nabla_\mu v_{\nu\rho}\nabla^\mu v^{\nu\rho} - 2Rv^2\Big) \\
&+\tfrac{1}{3}F^I(W)^{\mu\nu}\Big(\tfrac{1}{3}v_{\mu\nu}v^2 + 4v_{\mu\rho}v^{\rho\lambda}v_{\nu\lambda}\Big) + 4M^I v^{\mu\nu}v_{\nu\rho}v^{\rho\sigma}v_{\sigma\mu} + \tfrac{2}{3}M^I\epsilon_{\mu\nu\rho\sigma\lambda}v^{\mu\nu}v^{\rho\sigma}\nabla_\tau v^{\lambda\tau}\Big]
\end{aligned}$$

# Geometric invariants & their evaluation on squashed $S^5$:

**EA:** $$\mathrm{i}W = \frac{\mathrm{i}}{8\pi^2}\left(\frac{\kappa_1}{360}\mathbf{I_1} + \frac{\kappa_2 - \frac{3}{2}\kappa_2}{144}\mathbf{I_2} - \frac{\kappa_3}{24}\mathbf{I_3} - \frac{\kappa_{\mathrm{f}}^{G_{\mathrm{f}}}}{24}\mathbf{I_4^{G_f}}\right) + \mathcal{O}(\beta^0, \log\beta)$$

$$\begin{aligned}
I_2 &\equiv \int_{\mathcal{M}_5} A \wedge \mathrm{tr}\,(R \wedge R) + \text{ SUSY completion} \\
&\equiv 4\Big(\text{SUSY Euclidean FWW action in terms of } \mathcal{SW} \oplus \mathcal{V}_{\mathrm{KK}}\Big) \\
&\quad + \frac{2}{3}\Big(\text{SUSY Euclidean FRR action in terms of } \mathcal{P}_{\mathrm{KK}}\Big) \\
&= -\frac{2}{3}\frac{(\omega_1+\omega_2+\omega_3)^2}{\omega_1\omega_2\omega_2}(2\pi)^3 m_{\mathrm{KK}} \\
&\quad + 4\left[\frac{\omega_1^2+\omega_2^2+\omega_3^2}{2\omega_1\omega_2\omega_3} - \frac{(\omega_1+\omega_2+\omega_3)^2}{6\omega_1\omega_2\omega_3}\right](2\pi)^3 m_{\mathrm{KK}} \\
&= \frac{(2\pi)^3}{2}\frac{(\omega_1+\omega_2+\omega_3)^2}{\omega_1\omega_2\omega_3} m_{\mathrm{KK}}
\end{aligned}$$

# Summary:

$$\mathrm{i}W = -\log \mathcal{Z} = \frac{\mathrm{i}}{8\pi^2}\left(\frac{\kappa_1}{360} I_1 + \frac{\kappa_2 - \frac{3}{2}\kappa_3}{144} I_2 - \frac{\kappa_3}{24} I_3 - \frac{\kappa_{\mathrm{f}}^{G_{\mathrm{f}}}}{24} I_4^{G_{\mathrm{f}}}\right) + \mathcal{O}(\beta^0, \log\beta)$$

where

$$m_{\mathrm{KK}} = \frac{2\pi\mathrm{i}}{\beta},$$

we find:

$$I_1 = \int_{\mathcal{M}_5} A \wedge \mathrm{d}A \wedge \mathrm{d}A + \text{SUSY completion} = \frac{(2\pi)^3}{\omega_1\omega_2\omega_3}\left(\frac{2\pi\mathrm{i}}{\beta}\right)^3$$

$$I_2 = \int_{\mathcal{M}_5} A \wedge \mathrm{tr}\,(R \wedge R) + \text{SUSY completion} = -2(2\pi)^3 \frac{(\omega_1^2 + \omega_2^2 + \omega_3^2)}{\omega_1\omega_2\omega_3}\left(\frac{2\pi\mathrm{i}}{\beta}\right)$$

$$I_3 = \int_{\mathcal{M}_5} A \wedge \mathrm{Tr}\,(F_R \wedge F_R) + \text{SUSY completion} = \frac{(2\pi)^3}{2}\frac{(\omega_1 + \omega_2 + \omega_3)^2}{\omega_1\omega_2\omega_3}\left(\frac{2\pi\mathrm{i}}{\beta}\right)$$

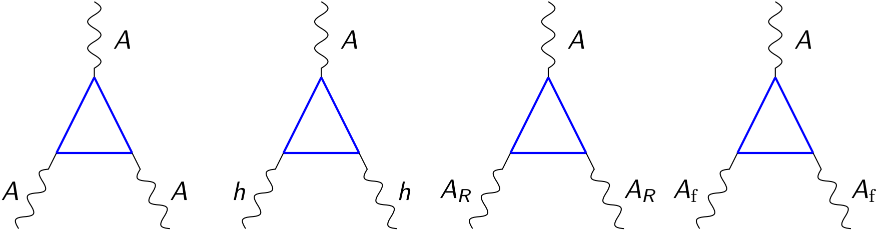
$$I_4 = \int_{\mathcal{M}_5} A \wedge \mathrm{Tr}\,(F_{G_{\mathrm{f}}} \wedge F_{G_{\mathrm{f}}}) + \text{SUSY completion} = \frac{(2\pi)^3}{\omega_1\omega_2\omega_3}\mu_{\mathrm{f}}^2\left(\frac{2\pi\mathrm{i}}{\beta}\right)$$

$$\log \mathcal{Z}_{S^1_\beta \times S^5} = -\frac{\pi}{\omega_1\omega_2\omega_3}\left[\frac{\kappa_1}{360}\left(\frac{2\pi}{\beta}\right)^3 + \frac{(\omega_1^2 + \omega_2^2 + \omega_3^2)(\kappa_2 - 3\kappa_3/2)}{72}\left(\frac{2\pi}{\beta}\right) + \frac{(\omega_1 + \omega_2 + \omega_3)^2\kappa_3}{48}\left(\frac{2\pi}{\beta}\right) + \frac{\mu_{\mathrm{f}}^2\kappa_{\mathrm{f}}^{G_{\mathrm{f}}}}{24}\left(\frac{2\pi}{\beta}\right)\right] + \mathcal{O}(\beta^0, \log\beta),$$

# KK reduction of free fields

- Assume weakly coupled phase in the EFT (e.g. Higgs branch)
- 1-loop exact

$A \wedge \mathrm{d}A \wedge \mathrm{d}A$ $\quad A \wedge \mathrm{tr}\,(\mathcal{R} \wedge \mathcal{R})$ $\quad A \wedge \mathrm{tr}\,(F_R \wedge F_R)$ $\quad A \wedge \mathrm{tr}\,(F_\mathrm{f} \wedge F_\mathrm{f})$




- Internal legs either massive 2-forms or massive fermions.
- Sum over KK tower contributions $\rightsquigarrow \kappa_i$

# $\kappa_i$ from KK reduction of effective action

From KK-reduction of (anti)chiral fermions $\psi_\pm$ and self-dual 2-forms, $B$ [Bonetti-Grimm-Hohenegger]:

$$
\begin{aligned}
\mathcal{I}^n_{\psi_-} &= \frac{1}{48\pi^2} n^3 I_1 + \frac{1}{384\pi^2} n I_2 \,, \\
\mathcal{I}^n_{\psi_+} &= -\frac{1}{48\pi^2} n^3 I_1 - \frac{1}{384\pi^2} n I_2 \,, \\
\mathcal{I}^n_{B} &= -\frac{4}{48\pi^2} n^3 I_1 + \frac{8}{384\pi^2} n I_2 \,,
\end{aligned}
$$

Thus, for 6d $\mathcal{N}=(1,0)$ supermultiplets ($T \to \left(B^-_{\mu\nu}, \phi, 2\psi^-\right)_{5d}$, tensor multiplet, $V \to (A_\mu, 2\psi^+)_{5d}$ vector multiplet, $H \to (4\phi, 2\psi^-)_{5d}$ hypermultiplet)

$$
\begin{aligned}
\mathcal{I}_T &= \frac{2-4}{48\pi^2}\frac{1}{120} I_1 - \frac{2+8}{384\pi^2}\frac{1}{12} I_2 - \frac{2}{32\pi^2}\frac{1}{12} I_3 \\
\mathcal{I}_V &= \frac{-2}{48\pi^2}\frac{1}{120} I_1 - \frac{-2}{384\pi^2}\frac{1}{12} I_2 - \frac{-2}{32\pi^2}\frac{1}{12} I_3 \\
\mathcal{I}^{n_H}_H &= \frac{2}{48\pi^2}\frac{1}{120} I_1 - \frac{2}{384\pi^2}\frac{1}{12} I_2 - \frac{2}{32\pi^2}\frac{1}{12} I_4^{USp(2n_H)}
\end{aligned}
$$

# Contributions from KK reduction

Thus, we conclude that from KK reduction of massless 6d fields in the weakly coupled phase (vortex-sheets do not contribute local CS terms)

$$\mathrm{i}W = \frac{\mathrm{i}}{8\pi^2}\left(\frac{n_T + n_V - n_H}{360} I_1 + \frac{n_H + 5n_T - n_V}{288} I_2 + \frac{n_T - n_V}{24} I_3 + \frac{1}{24} I_4^{USp(2n_H)}\right) + \mathcal{O}(\beta^0, \log\beta)\,.$$

# 4d $\mathcal{N} = 2$ examples

**Higgs branch EA of 4d $\mathcal{N} = 2$ theories:**

(i) {free ml hypermultiplets} $:=$ pure Higgs branch.

(ii) {free ml hypermultiplets} $\oplus$ {free ml $U(1)$ vector multiplets}.

(iii) {mixed branch} $\oplus$ {decoupled interacting SCFT (with trivial Higgs branch)}.

**Examples:**

(i) AD theories (non-Lagrangian)

a) $(A_1, A_{2n+1})$: $\dim \mathrm{CB} = n$, $\dim_{\mathbb{H}} \mathrm{HB} = 2$, and $c - a = \frac{1}{24}$ ✓

b) $(A_1, D_{2n+2})$: $\dim \mathrm{CB} = n$, $\dim_{\mathbb{H}} \mathrm{HB} = 1$, and $c - a = \frac{2}{24}$ ✓

(ii) Lagrangian theories: $\dim \mathrm{CB} = n_{\mathrm{v}}$, $\dim_{\mathbb{C}} \mathrm{HB} = n_{\mathrm{h}}$, and $c - a = \frac{n_{\mathrm{h}} - n_{\mathrm{v}}}{24}$ ✓

(iii) $(A_1, D_{2n+1}) \rightsquigarrow (A_1, A_{2n-2}) \oplus$ {free hyper}:

$$(c-a)[(A_1, D_{2n+1})] = (c-a)[(A_1, A_{2n-2})] + \underbrace{(c-a)[\textit{free hyper}]}_{=1/24} \quad ✓$$

# Global anomalies from CS EA

$\exists$ direct relation between the CS levels $\kappa_i$ and the phases from global anomalies!

- Large (bg) diffeomorphism on $S^1 \times \mathcal{M}_5$ (*e.g.* $\mathcal{M}_5 = S^1_{x^5} \times \mathcal{M}_4$)

$$\tau \;\rightarrow\; \tau + \frac{n\beta}{2\pi R_5} x^5$$

where $n \in \mathbb{Z}$ (preserve the bc for the fermionic dofs along the $S^1_{x^5}$).

$\Leftrightarrow$ background large gauge transformation of the graviphoton $A^4$

$$A \;\rightarrow\; A + \frac{n}{R_5}\mathrm{d}x^5$$

- Theories with (mixed) gravitational anomalies, the $Z$ not invariant under such a large bg diffeomorphism:

$$Z[A + \delta A] \;=\; e^{-\mathrm{i}\pi\eta} Z[A]\,.$$

- Global gravitational anomalies $\leftrightsquigarrow$ anomalies under large gauge transformations

[4] Recall our metric is $\mathrm{d}s_6^2 = \left(\mathrm{d}\tau + \frac{\beta}{2\pi} A_i \mathrm{d}x^i\right)^2 + h_{ij}\mathrm{d}x^i \mathrm{d}x^j$.

# Global anomalies from CS EA

- The 5d CS EA completely captures this anomalous diffeomorphism!
- Under

$$A \to A + \frac{n}{R_5}\mathrm{d}x^5$$

the effective action transforms as

$$\delta W = n\int_{\mathcal{M}_4}\left(\frac{\kappa_1}{480\pi}\mathrm{d}A\wedge\mathrm{d}A - \frac{\pi}{72}(\kappa_2 - \frac{3}{2}\kappa_3)p_1 - \frac{\pi}{12}\kappa_3 c_2(SU(2)_R) - \frac{\pi}{12}\kappa_{\mathrm{f}}^{G_{\mathrm{f}}}c_2(G_{\mathrm{f}})\right)$$

- The integral satisfies quantization conditions on the spin manifold $\mathcal{M}_4$

$$m_1 \equiv \frac{1}{2(2\pi)^2}\int_{\mathcal{M}_4}\mathrm{d}A\wedge\mathrm{d}A \in \mathbb{Z}, \qquad m_2 \equiv \frac{1}{24}\int_{\mathcal{M}_4}p_1 \in 2\mathbb{Z}$$

$$m_3 \equiv \int_{\mathcal{M}_4}c_2(SU(2)_R) \in \mathbb{Z}, \qquad m_{\mathrm{f}} \equiv \int_{\mathcal{M}_4}c_2(G_{\mathrm{f}}) \in \mathbb{Z}$$

- We find that the anomalous phase in $Z[A+\delta A] = e^{-\mathrm{i}\pi\eta}Z[A]$ is given by

$$\eta = \frac{nm_1}{60}\kappa_1 + \frac{2nm_2}{3}(\kappa_2 - \frac{3}{2}\kappa_3) - \frac{n}{12}(m_3\kappa_3 + m_{\mathrm{f}}\kappa_{\mathrm{f}}^{G_{\mathrm{f}}}) \mod 2$$

## I. $\kappa_i$ related to $\eta$:

It follows that $\kappa_i$ are fixed (up to jumps). Supports our argument that $\kappa_i = \mathrm{cst}$ along flows.

## II. Global gravitational anomalies given by $\eta$-invariant on Higgs branch:

We prove that $\kappa_i = \mathrm{cst}$ on the Higgs branch, and that $\kappa_i$ are related to global gravitational anomalies. Thus, we prove that on the Higgs branch only the $\eta$-invariant contributes to global gravitational anomalies. ($\eta$-invariant measures fermionic global anomalies, see *e.g.* [*Yonekura-Witten*])

## III. $a$ vs. $c$-anomalies

- 6d: $\kappa_i = \mathrm{cst}$ on Higgs branch implies that going on the Higgs branch there are relations between $a$ and $c_i$ anomalies. [*Cordova-Dumitrescu-Intriligator; ...*].
- 4d: $\kappa = c - a = \mathrm{cst}$ on Higgs branch $\Longrightarrow$ $\Delta a = \Delta c$ on Higgs branch. Follows also from anomaly-matching formula by [*Shapere-Tachikawa*].

## IV. Higher-derivative terms are geometric invariants

5d higher derivative terms (Chern-Simons terms $I_j$, $j = 1, \ldots, 4$) are geometric invariants only dependent on the THF of $\mathcal{M}_5$ (also independent of choice of gauge-fixing). Proven for $I_{\mathrm{FFF}}$ and $I_{\mathrm{FF_fF_f}}$ (vm action) [*unpublished*].

## Brief primer:

- "New Cardy limit" in 4d/6d [*Choi-Kim$^2$-Nahmgoong; Cabo-Bizet-Cassani-Martelli-Murthy; Ardehali; Kim$^2$-Song; Nahmgoong; ...*] $\leftrightarrow$ black hole microstates in dual sugra (in large $N$ and $\omega_i << 1$ limit)[5]
- **New features:** Complex fugacities (complex SUSY background/different spin structure; see also [*Chang-MF-Lin-Wang*]) $\rightsquigarrow$ complex saddle points $\rightsquigarrow$ additional leading order term $\propto (5a - 3c) \neq 0$ for holographic theories.
- **Different limit:** instead of $\omega_i^{(\mathrm{there})} \to 0$, one takes $\omega_i^{(\mathrm{there})} \to 0 + \mathrm{i}(\cdots)$ $\rightsquigarrow$ phase transitions at large $N$! [*Cabo-Bizet-Murthy; Ardehali-Hong-Liu*]

[5] $\beta\omega_1^{(\mathrm{here})} = \omega_1^{(\mathrm{there})}$, $\beta\omega_2^{(\mathrm{here})} = \omega_2^{(\mathrm{there})}$, $\beta\omega_3^{(\mathrm{here})} = 2\pi\mathrm{i} + \omega_3^{(\mathrm{there})}$.

# Some corollaries II:

V. "New Cardy limits" (for simplicity 4d)

**"Effective action approach":**

- **3d effective action is different:**

$$W^{\text{theirs}} \sim \mathrm{CS}^{\text{gauge-invariant}}_{\text{ours},\cancel{\text{SUSY}}} + \underbrace{W^{\text{gauge-non-invariant}}_{\cancel{\text{SUSY}}}}_{\text{related to \textbf{perturbative} anomalies}}$$

$$\sim \text{non-SUSY thermal effective action}$$

$$W^{\text{ours}} \sim \underbrace{\mathrm{CS}^{\text{gauge-invariant}}_{\text{ours},\text{SUSY}}}_{\text{related to \textbf{global} anomalies}}$$

- Arguments based on "thermal effective action" [*Jensen-Loganayagam-Yarom*] (supersymmetrized?)
- **Question:** Would expect that $\mathrm{CS}^{\text{gauge-invariant}}_{\text{ours},\text{SUSY}} + W^{\text{gauge-non-invariant}}_{\text{SUSY}}$ gives the right answer?
- Missing SUSY completion of "thermal effective action", *i.e.* $W^{\text{gauge-non-invariant}}_{\text{SUSY}} =$???

However our general strategy should provide proof!

# Future directions

I. Relation to modularity of 4d/6d index [*Dedushenko-MF*]: high-low temperature relation, *e.g.* Casimir vs Cardy.

II. Understand contributions from BPS strings and BPS states to the Cardy formula.

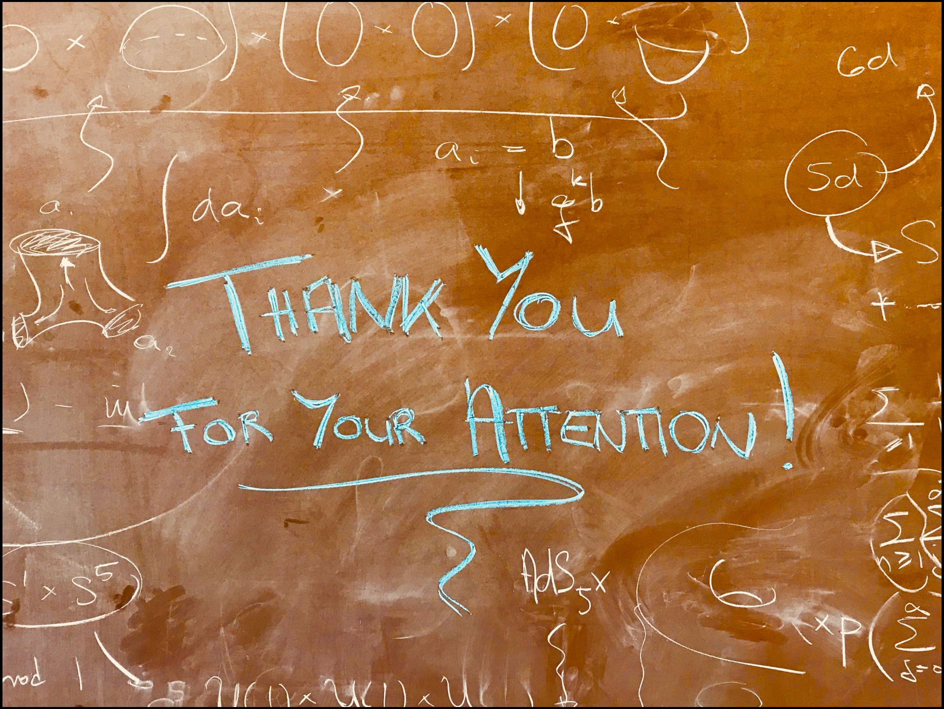
III. Proving “new" $\mathbb{C}$-Cardy limit in 4d/6d [*Choi-Kim$^2$-Nahmgoong; Cabo-Bizet-Cassani-Martelli-Murthy; Kim$^2$-Song; Nahmgoong; ...*] from effective field theory and relation to black hole microstates.

IV. Cardy limits in other dimensions from effective field theory and relation to global anomalies?

V. Relation between Cardy limits in different dimensions from dimensional reduction and $d$-anomalies vs $d-1$ anomalies (*e.g.* mixed higher form symmetries, *etc*)

VI. Holographic dual of effective action? Higher derivative terms in 6d vs counter-terms. Measurable effect of superconformal anomalies, $\cdots$

VII. Higher-derivative counter-terms and corrections to black hole entropy in $AdS_5$?


THANK YOU
FOR YOUR ATTENTION!