Extremal Black Holes and EFTs

Grant Remmen

Berkeley Center for Theoretical Physics
Miller Institute for Basic Research in Science
University of California, Berkeley

19XX.XXXXX with Nima Arkani-Hamed
1903.09156, 1801.08546 with Cliff Cheung & Junyu Liu

University of Michigan, November 2019

BERKELEY CENTER FOR THEORETICAL PHYSICS

MILLER INSTITUTE

The space of effective field theories

The space of EFTs

How do we build a quantum field theory?

- Write down a Lagrangian, built out of operators $\mathcal{O}_i$, with couplings c_i:

$$\mathcal{L} = \overline{\mathcal{L}} + \sum_i c_i \mathcal{O}_i$$

  and then just quantize it.

The space of EFTs

How do we build a quantum field theory?

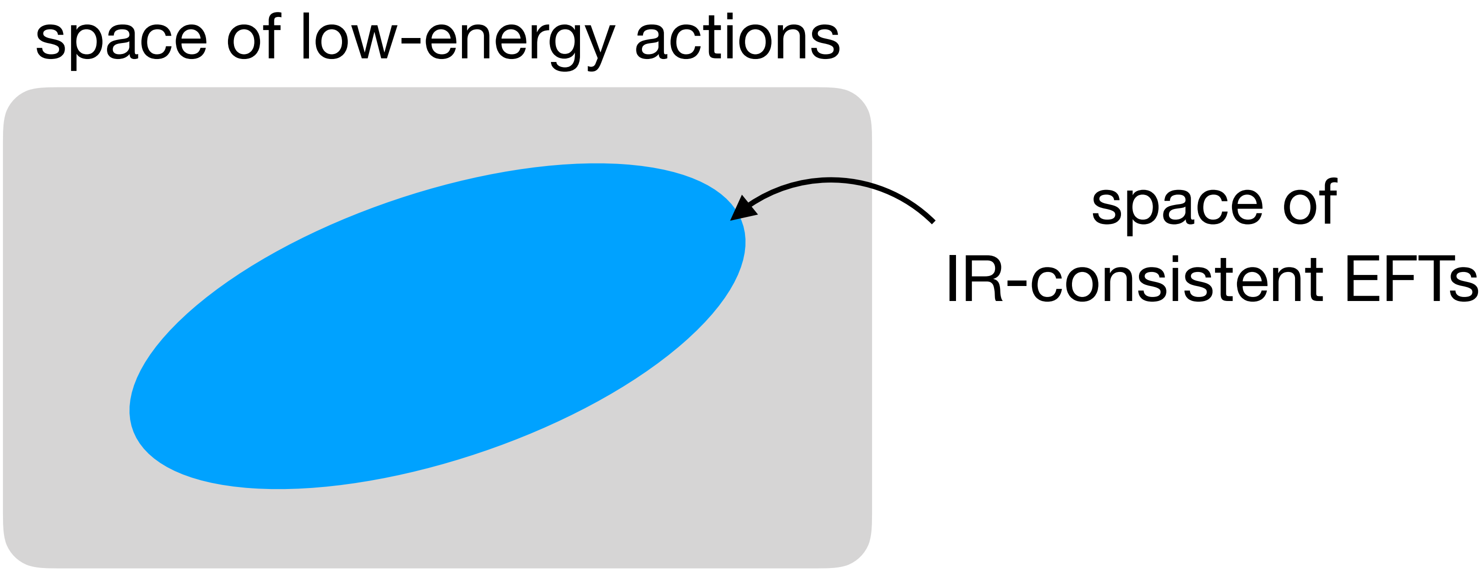
- Write down a Lagrangian, built out of operators $\mathcal{O}_i$, with couplings c_i:

$$\mathcal{L} = \overline{\mathcal{L}} + \sum_i c_i \mathcal{O}_i$$

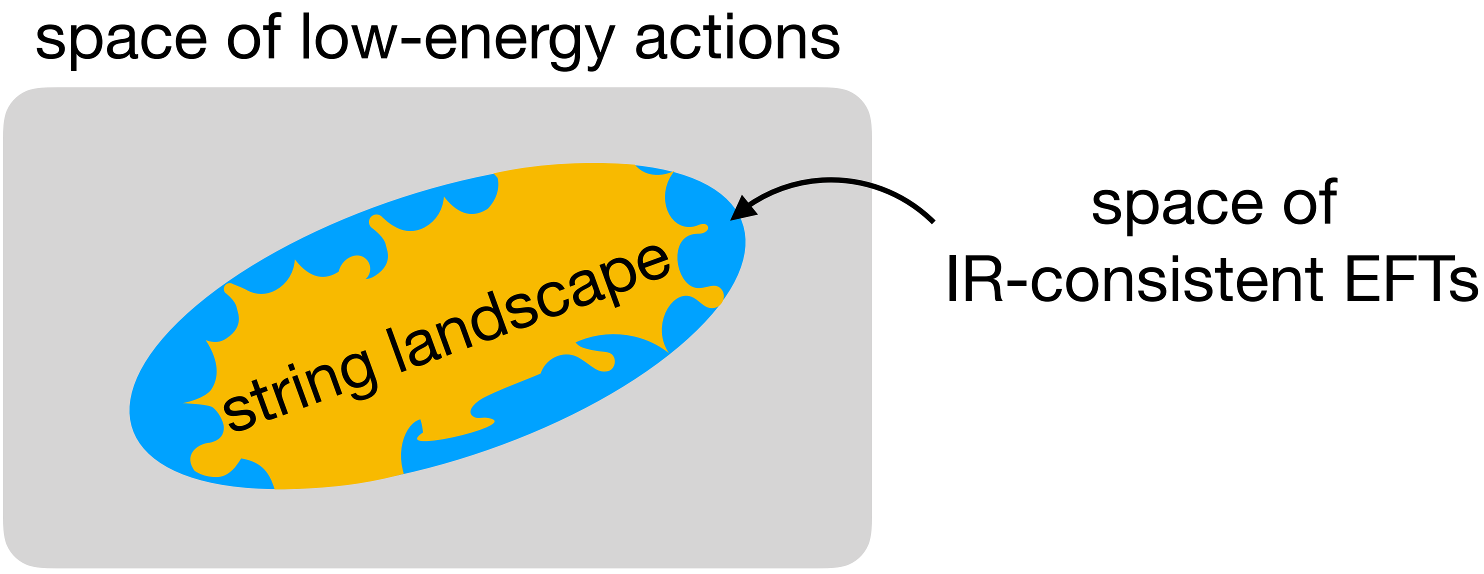
  and then just quantize it.

- Is this guaranteed to create a consistent EFT? No! Not all couplings c_i are allowed
- Certain signs of couplings violate infrared physics principles:
  - Unitarity
  - Causality
  - Analyticity
  - Examples:
    - Einstein-Maxwell theory Cheung, GR [1407.7865]; Cheung, Liu, GR [1801.08546, 1903.09156]; Bellazzini, Lewandowski, Serra [1902.03250]
    - Higher-curvature gravity (R^2, R^4 terms) Bellazzini, Cheung, GR [1509.00851]; Cheung, GR [1608.02942]
    - Massive gravity Cheung, GR [1601.04068]
    - $(\partial\phi)^4$ and F^4 couplings Adams et al. [hep-th/0602178]
    - Higher-point couplings Chandrasekaran, GR, Shahbazi-Moghaddam [1804.03153]
    - Standard Model EFT GR, Rodd [1908.09845]
    - a-theorem Komargodski, Schwimmer [1107.3987]; Elvang et al. [1205.3994]

The Swampland Program



The Swampland Program



In addition to IR constraints, a top-down approach:

- Build consistent vacua in quantum gravity
- Map out the space of possible low-energy EFTs that result
- Not tractable to do this exhaustively
- String landscape vs. swampland

Is it possible that the full landscape can be delineated from the swampland using IR consistency alone?

Black hole kinematics and the WGC

The Weak Gravity Conjecture

- An ultraviolet consistency condition for quantum gravity.

- Statement: For any $U(1)$ gauge theory coupled consistently with quantum gravity, there must exist in the spectrum a state with charge q and mass m such that

$$\boxed{\frac{q}{m} > \frac{1}{m_{\mathrm{Pl}}}}$$

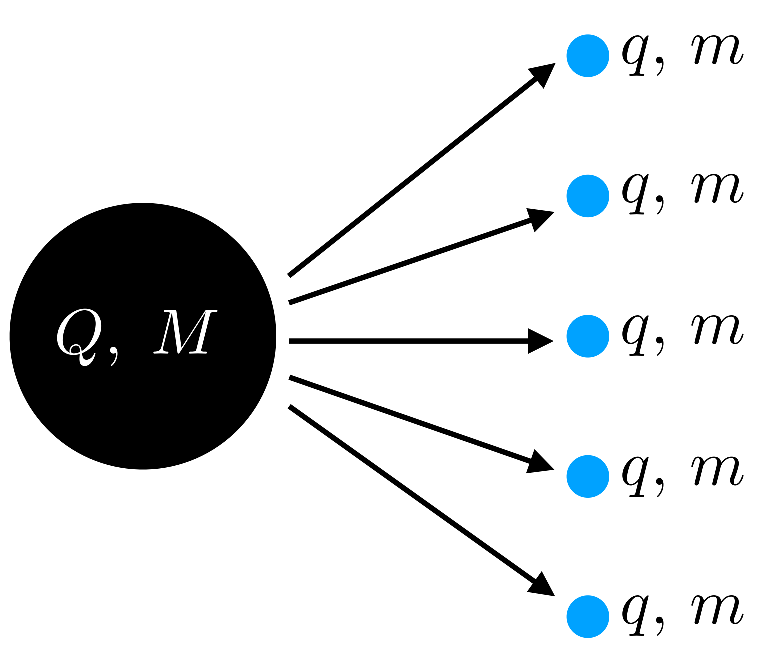
- Thus, “gravity is the weakest force”.

The Weak Gravity Conjecture

- An ultraviolet consistency condition for quantum gravity.
- Statement: For any $U(1)$ gauge theory coupled consistently with quantum gravity, there must exist in the spectrum a state with charge q and mass m such that

$$\boxed{\frac{q}{m} > \frac{1}{m_{\rm Pl}}}$$

- Thus, “gravity is the weakest force”.



Original justification: Arkani-Hamed et al. [hep-th/0601001]

A black hole of charge Q and mass M can only decay into states satisfying

$$\frac{q}{m} > \frac{Q}{M}$$

Extremal BH decay $\Longrightarrow$ WGC

Why BH decay? BH remnant pathologies

Charges

Charged black hole in $D = 4$ spacetime dimensions:
electric charge Q, magnetic charge $\widetilde{Q}$, mass M, angular momentum J
measured asymptotically (ADM quantities)

Natural units:

$$m = \frac{\kappa^2 M}{8\pi} \qquad q = \frac{\kappa Q}{4\sqrt{2}\pi} \qquad a = \frac{J}{M}$$

$$\kappa^2 = 8\pi G \qquad \widetilde{q} = \frac{\kappa \widetilde{Q}}{4\sqrt{2}\pi}$$

Extremality parameter:

$$\xi = \frac{\sqrt{m^2 - (q^2 + \widetilde{q}^2 + a^2)}}{m^2}$$

$\xi = 0 \Longrightarrow$ extremal
$\xi = 1 \Longrightarrow$ Schwarzschild

Dyon parameter:

$$\mu = \frac{q^2 - \widetilde{q}^2}{q^2 + \widetilde{q}^2}$$

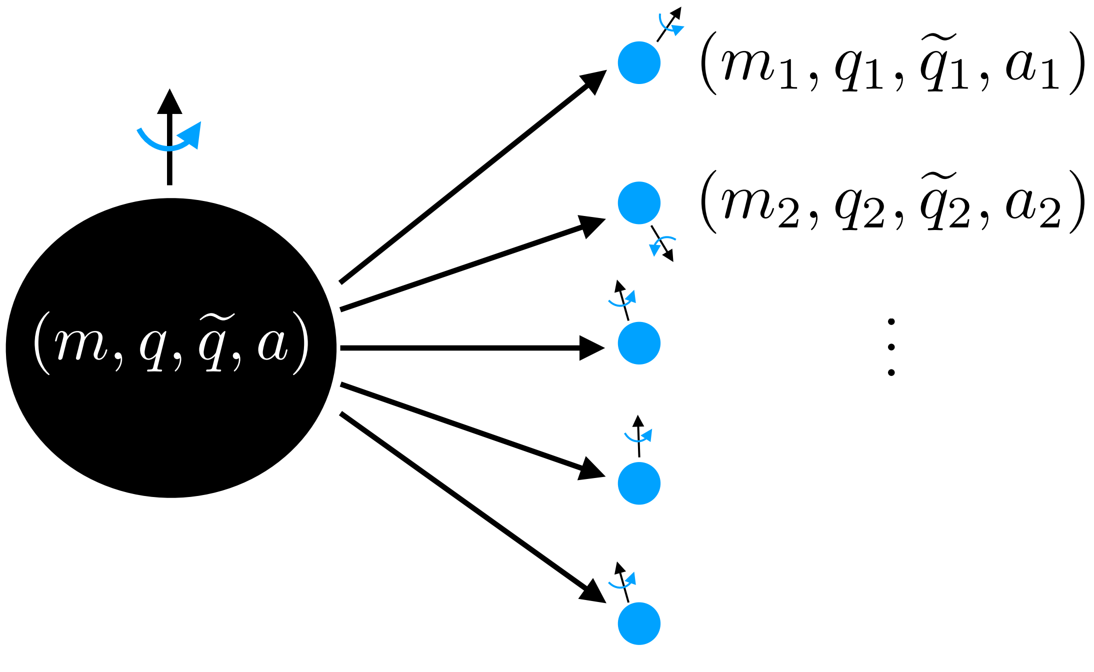
$\mu = +1 \Longrightarrow$ electric
$\mu = -1 \Longrightarrow$ magnetic

Spin parameter:

$$\nu = \frac{a}{r_{\mathrm{H}}}$$

$\nu = 0 \Longrightarrow$ nonspinning RN
$\nu = \sqrt{\frac{1-\xi}{1+\xi}} \Longrightarrow$ neutral Kerr

Black hole decay kinematics



- Consider a dyonic, spinning black hole with parameters $(m, q, \widetilde{q}, a)$ decaying to lighter states i with parameters $(m_i, q_i, \widetilde{q}_i, a_i)$, where the angular momenta are $j_i = 8\pi m_i a_i / \kappa^2$ (including both spin and orbital)

- Charge conservation:

$$\sum_i (q_i, \widetilde{q}_i) = (q, \widetilde{q})$$

- Decay products need some kinetic energy, so energy conservation gives:

$$\sum_i m_i < m$$

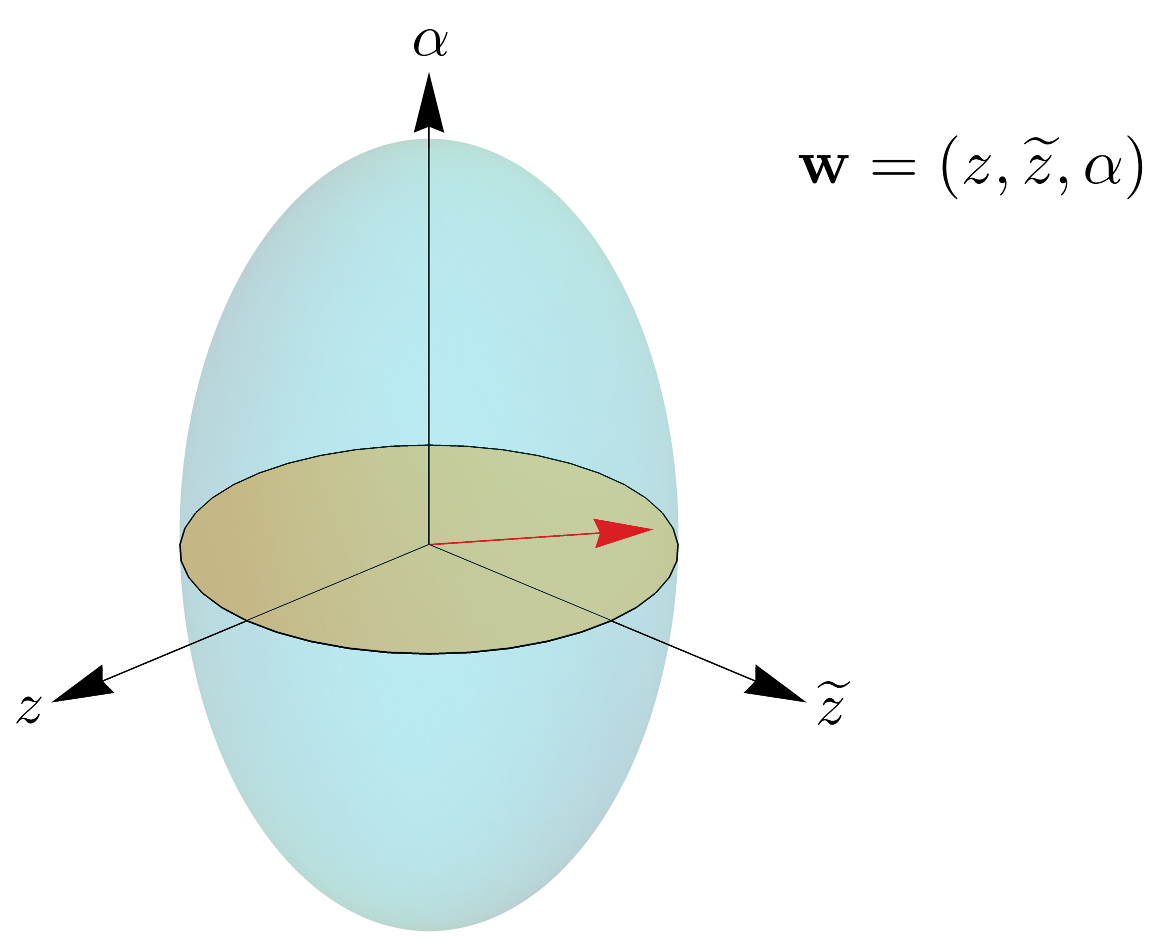
- Spins might be misaligned, so angular momentum conservation gives:

$$J = aM \leq \sum_i j_i$$

Black hole decay kinematics

- Unitless ratios: $z = q/m$
  $\tilde{z} = \tilde{q}/m$
  $\alpha = am\kappa^{-2} = J/8\pi$
- Physical KN black holes form spheroid:

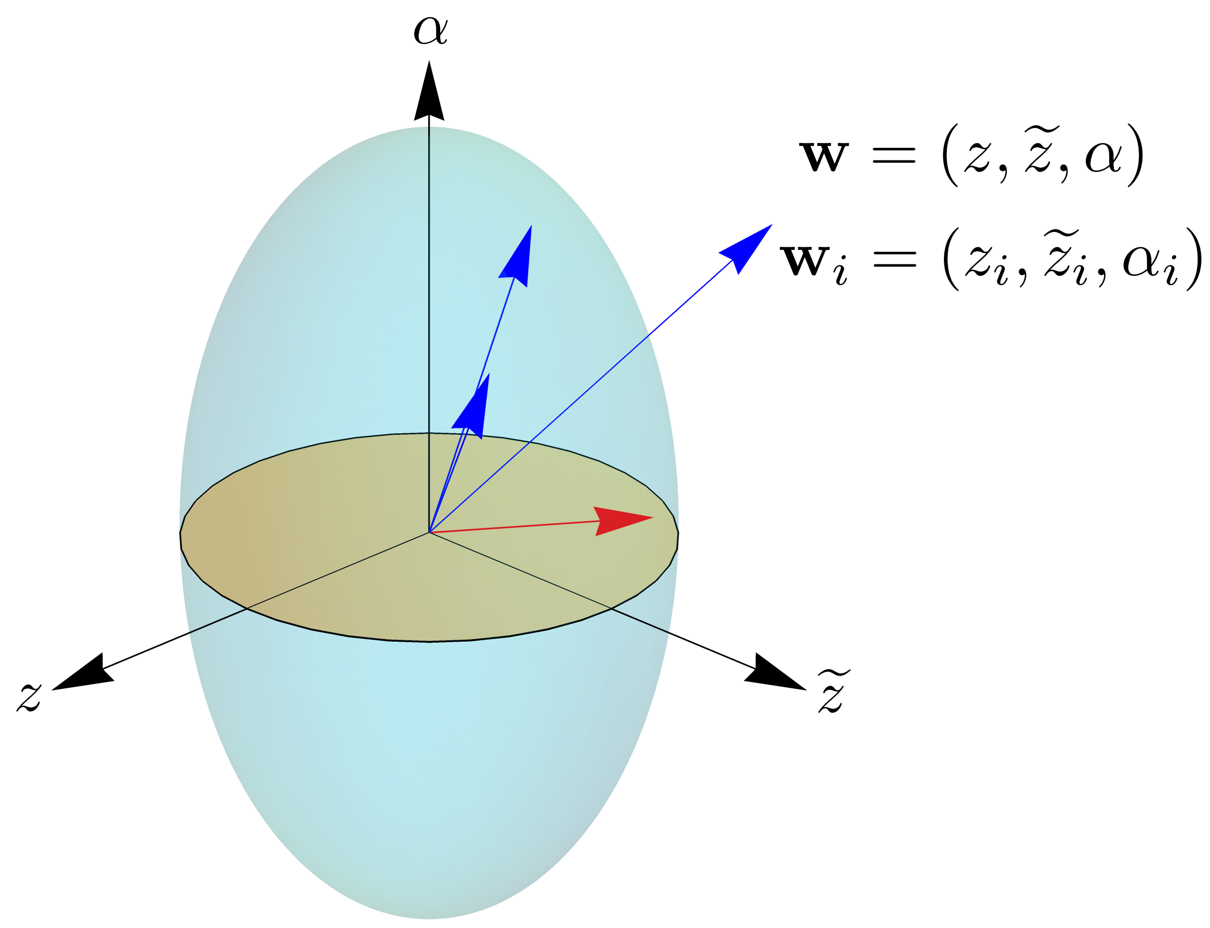
$$\zeta^2 = z^2 + \tilde{z}^2 + (\alpha^2\kappa^4/m^4) \leq 1$$



Black hole decay kinematics

- Unitless ratios:
$$z_i = q_i/m_i$$
$$\tilde{z}_i = \tilde{q}_i/m_i$$
$$\alpha_i = a_i m \kappa^{-2} = j_i/8\pi\sigma_i$$
$$\sigma_i = m_i/m$$

- Decay requires $\sum_i \sigma_i < 1$, $\sum_i (\sigma_i z_i, \sigma_i \tilde{z}_i) = (z, \tilde{z})$, $\alpha \leq \sum_i \sigma_i \alpha_i$

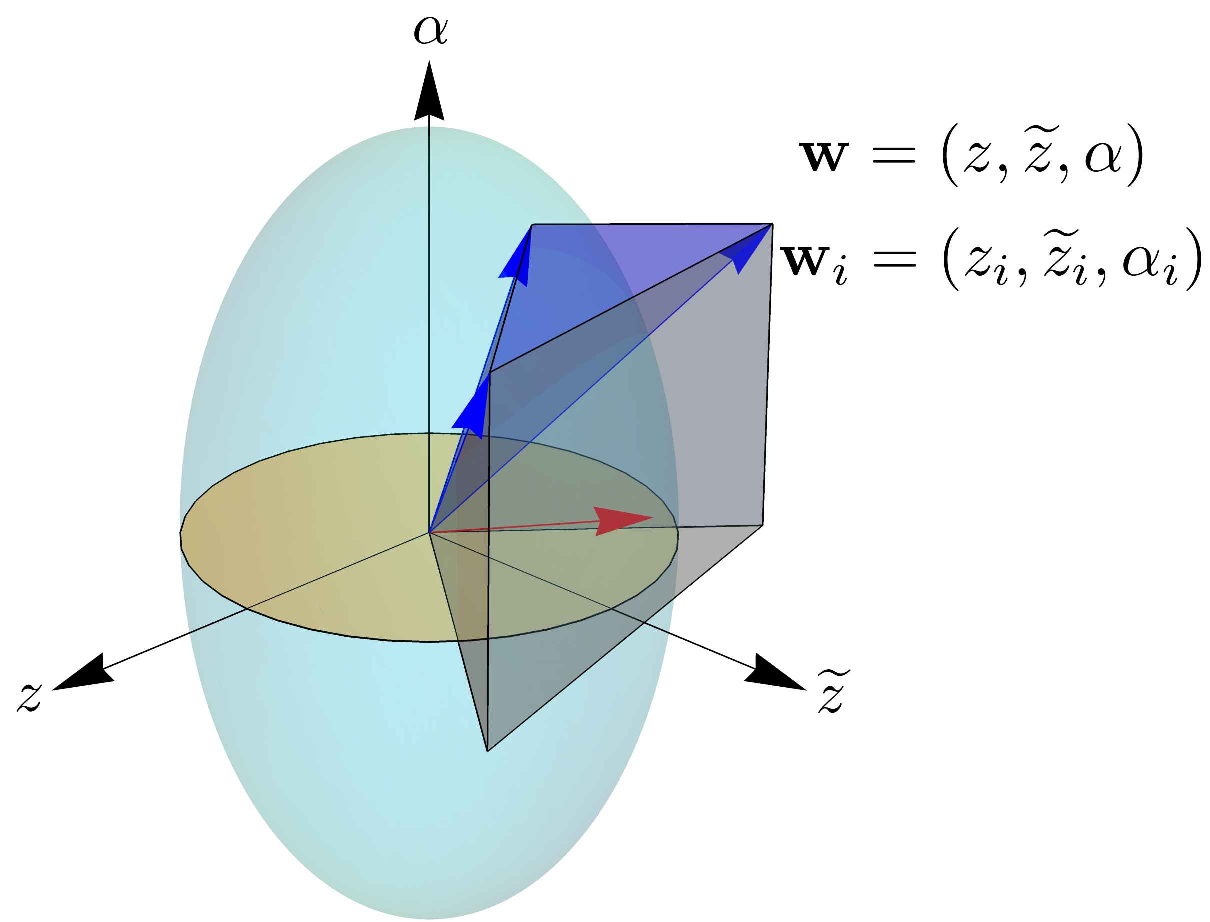


Black hole decay kinematics

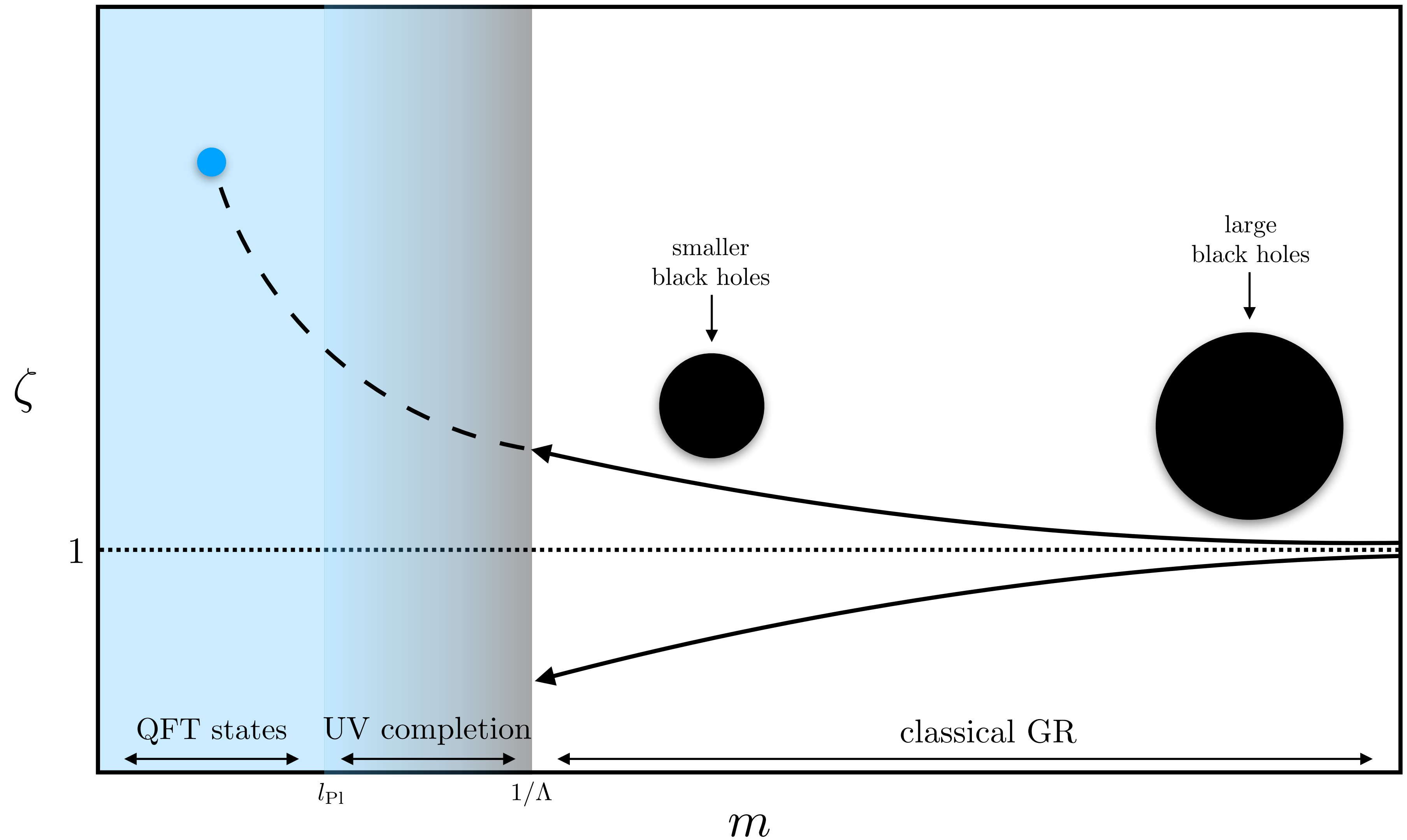
- Unitless ratios:
$$z_i = q_i/m_i$$
$$\tilde{z}_i = \tilde{q}_i/m_i$$
$$\alpha_i = a_i m \kappa^{-2} = j_i/8\pi\sigma_i$$
$$\sigma_i = m_i/m$$

- Decay requires $\sum_i \sigma_i < 1$, $\sum_i(\sigma_i z_i, \sigma_i \tilde{z}_i) = (z, \tilde{z})$, $\alpha \leq \sum_i \sigma_i \alpha_i$

- $\mathbf{w}$ must either inside convex hull of $\mathbf{w}_i$ or between hull and $\alpha = 0$ plane



The Weak Gravity Conjecture



Corrections to Einstein and Maxwell equations induced by $\Delta\mathcal{L}$ change the extremal values of $z \in [0, 1+\Delta z]$ (or ζ in the spinning case).

- If $\Delta z > 0$, then black holes themselves provide WGC states.
- If $\Delta\zeta > 0$, then spinning charged black holes can decay directly to other spinning, charged black holes that are nearly at rest.

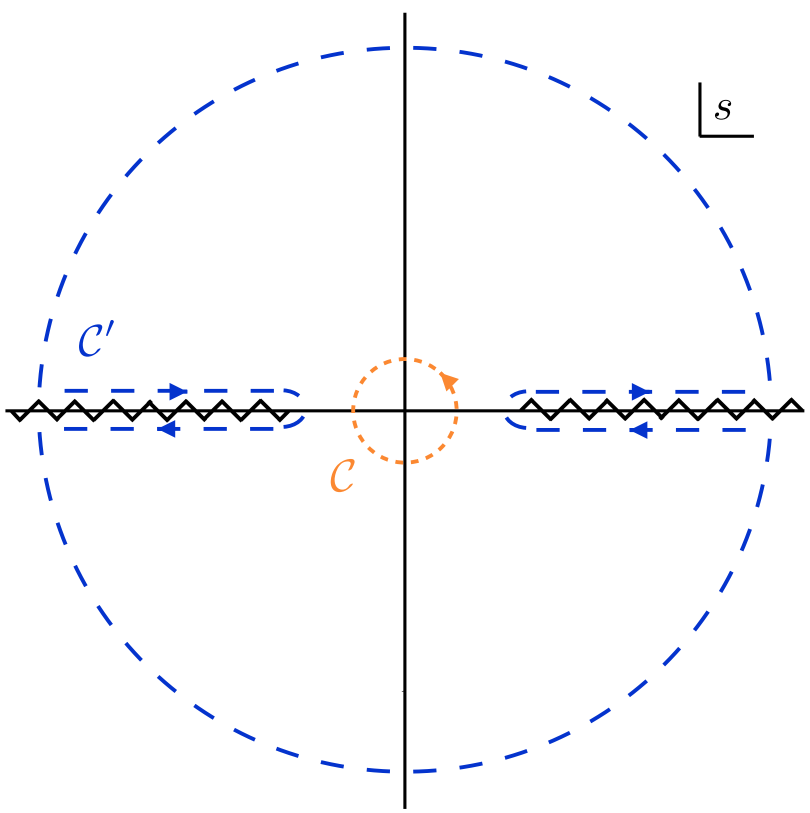
Positivity from infrared consistency

Positivity from scattering amplitudes

- Take forward scattering amplitude of four-point EFT operator with $4n$ derivatives, e.g.,$(\partial\phi)^4$: Adams et al. [hep-th/0602178]

$$\mathcal{A} = \lambda s^{2n}$$

- Contour integral around origin to extract Wilson coefficient:



$$\begin{aligned}
\lambda &= \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{\mathrm{d}s}{s^{2n+1}}\mathcal{A} && \text{[residue thm]}\\
&= \frac{1}{2\pi i}\oint_{\mathcal{C}'} \frac{\mathrm{d}s}{s^{2n+1}}\mathcal{A} && \text{[analyticity]}\\
&= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\mathrm{d}s}{s^{2n+1}}\mathrm{disc}\,\mathcal{A} && \text{[boundary} \to 0\text{]}\\
&= \frac{1}{i\pi}\int_{0}^{\infty} \frac{\mathrm{d}s}{s^{2n+1}}\mathrm{disc}\,\mathcal{A} && \text{[crossing sym]}\\
&= \frac{2}{\pi}\int_{0}^{\infty} \frac{\mathrm{d}s}{s^{2n+1}}\mathrm{Im}\,\mathcal{A} && \text{[Schwarz reflection]}\\
&= \frac{2}{\pi}\int_{0}^{\infty} \frac{\mathrm{d}s}{s^{2n}}\sigma(s) > 0 && \text{[optical thm]}
\end{aligned}$$

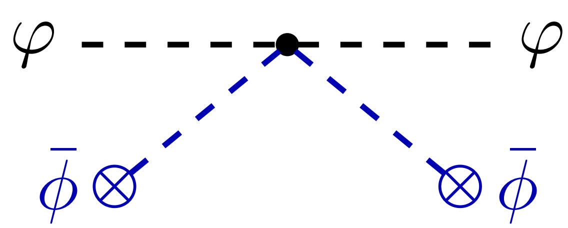
Positivity from scattering amplitudes

- Take forward scattering amplitude of four-point EFT operator with $4n$ derivatives, e.g.,$(\partial\phi)^4$: Adams et al. [hep-th/0602178]

$$\mathcal{A} = \lambda s^{2n}$$

- Contour integral around origin to extract Wilson coefficient: $\lambda > 0$

- Connection to causality: If $\lambda < 0$, perturbative signals are superluminal in nonzero classical backgrounds.



Graviton scattering

- Ideally, we would like to bound the leading four-derivative Einstein-Maxwell theory using analyticity of scattering.

$\overline{\mathcal{L}} = \frac{1}{2\kappa^2} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$, where $\mathcal{L} = \overline{\mathcal{L}} + \Delta\mathcal{L}$ and:

$$\begin{aligned}\Delta\mathcal{L} = \quad & c_1 R^2 + c_2 R_{\mu\nu} R^{\mu\nu} + c_3 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \\ & + c_4 R F_{\mu\nu} F^{\mu\nu} + c_5 R_{\mu\nu} F^{\mu\rho} F^{\nu}{}_{\rho} + c_6 R_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} \\ & + c_7 F_{\mu\nu} F^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} + c_8 F_{\mu\nu} F^{\nu\rho} F_{\rho\sigma} F^{\sigma\mu}\end{aligned}$$

- Obstruction: on-shell graviton exchange leads to t-channel singularity, $\mathcal{A} \sim G_N s^2/t$

- We will come back to four-derivative action later. For now, we move to higher-derivative actions that can be bounded from forward dispersion relation arguments. More derivatives $\Longrightarrow$ higher power in s $\Longrightarrow$ contour cancels s^2/t

Graviton scattering

- Instead, let us take a gravitational EFT where the first operators start at $\mathcal{O}(R^4_{abcd})$.

- Consistent as an EFT: At tree level, $\mathcal{O}(R^4_{abcd})$ terms will not contribute to two- or three-point couplings in the graviton.

- Possible motivation: type II string theory

- In four dimensions, there are two independent operators:

$$\boxed{\Delta\mathcal{L} = \frac{c}{\kappa^2\Lambda^6}(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})^2 + \frac{\widetilde{c}}{\kappa^2\Lambda^6}(R_{\mu\nu\rho\sigma}\widetilde{R}^{\mu\nu\rho\sigma})^2}$$

where $\widetilde{R}_{\mu\nu\rho\sigma} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}R^{\alpha\beta}{}_{\rho\sigma}$

Graviton scattering

- In four dimensions, there are two independent operators:

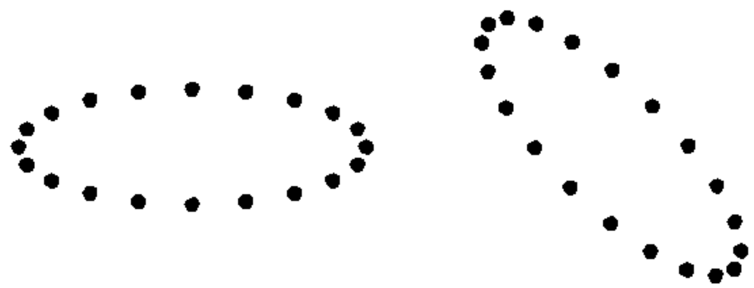
$$\Delta\mathcal{L} = \frac{c}{\kappa^2\Lambda^6}(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})^2 + \frac{\widetilde{c}}{\kappa^2\Lambda^6}(R_{\mu\nu\rho\sigma}\widetilde{R}^{\mu\nu\rho\sigma})^2$$

- Forward graviton-graviton scattering amplitude:

$$\mathcal{A}(s) = \frac{16\kappa^2 s^4}{\Lambda^6}(c\,\cos^2\theta + \widetilde{c}\,\sin^2\theta)$$

where

$$\epsilon_1 = \cos\theta_1\epsilon_+ + \sin\theta_1\epsilon_\times$$
$$\epsilon_2 = \cos\theta_2\epsilon_+ + \sin\theta_2\epsilon_\times$$
$$\theta = \theta_1 - \theta_2$$



- Analyticity and unitarity require $c > 0, \quad \widetilde{c} > 0$ Bellazzini, Cheung, GR [1509.00851]

- Can also reach this conclusion by demanding subluminality of gravitons in various backgrounds. Gruzinov, Kleban [hep-th/0612015]

(Riemann)^4 corrections to black holes

Arkani-Hamed, GR [19XX.XXXXX]

Deforming the RN solution

- When $J = 0$, we can use a spherically-symmetric ansatz:

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \mathrm{d}r^2/g(r) + r^2\mathrm{d}\Omega^2$$

- Use spherical symmetry to invert the Ricci tensor in terms of the metric coefficients Kats, Motl, Padi [hep-th/0606100]; Cheung, Liu, GR [1801.08546]

- Solve the perturbed Einstein equations to obtain corrected black hole solution:

$$f(r) = \bar{g}(r) - \frac{64m^3c}{715\Lambda^6 r^{14}} \times [2860(11m - 8r)r^4 + 572mr^3z^2(-208m + 141r) + 104m^2r^2z^4(1593m - 925r) + 520m^3rz^6(-163m + 77r) + 12705m^5z^8]$$

$$g(r) = \bar{g}(r) - \frac{64m^3c}{715\Lambda^6 r^{14}} \times [2860(67m - 36r)r^4 + 1716mr^3z^2(-521m + 250r) + 260m^2r^2z^4(5733m - 2266r) + 5720m^3rz^6(-185m + 49r) + 252945m^5z^8]$$

$$\bar{g}(r) = 1 - \frac{2m}{r} + \frac{q^2}{r^2}$$

Corrections to extremality

- As shown in Kats, Motl, Padi [hep-th/0606100]; Cheung, Liu, GR [1801.08546], deforming the black hole solution by higher-derivative operators changes the physical range of black hole charge-to-mass ratios (with no naked singularities):

$$z \in [0, 1 + \Delta z]$$

where

$$\Delta z = -\frac{g(\overline{r}_{\mathrm{H}}, 1)}{\partial_z \overline{g}(r, z)|_{\substack{r \to \overline{r}_{\mathrm{H}} \\ z \to 1}}} = -\frac{f(\overline{r}_{\mathrm{H}}, 1)}{\partial_z \overline{f}(r, z)|_{\substack{r \to \overline{r}_{\mathrm{H}} \\ z \to 1}}}$$

- For the $\mathrm{Riemann}^4$ correction, we find:

$$\boxed{\Delta z = +\frac{2208c}{715\Lambda^6 m^6} > 0}$$

Corrections to extremality

- For the spinning black hole, we no longer have spherical symmetry, so solving the higher-derivative Einstein equations is an open problem.

- However, Cheung, Liu, GR [1903.09156] derived the general relation between the extremality shift and on-shell shift in the action:

$$\Delta\zeta = \frac{\kappa^2}{8\pi m} \lim_{\zeta\to 1} \left(\int \mathrm{d}^3x \sqrt{-g}\,\Delta\mathcal{L}|_{\mathrm{KN}} \right)$$

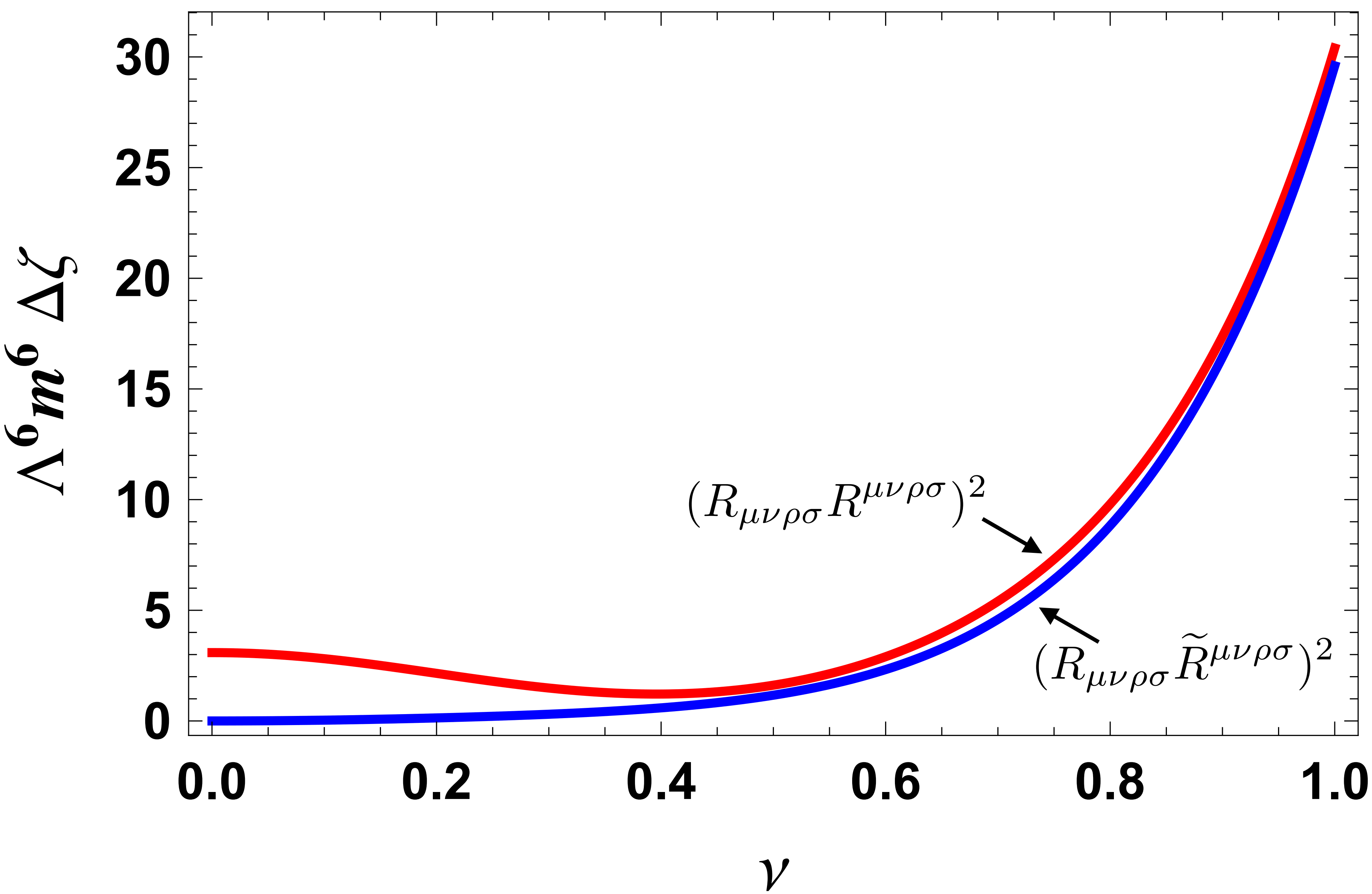
  Physical region of extremality parameter: $\zeta \in [0, 1+\Delta\zeta]$

- This follows from three facts:
  - The shift in the Wald entropy of a BH, induced by $\Delta\mathcal{L}$, equals its shift in Euclidean action Reall, Santos [1901.11535]
  - In the extremal limit, the entropy correction for BH with fixed charges is dominated by shift in horizon area Cheung, Liu, GR [1801.08546, 1903.09156]
  - Shift in horizon coordinate and extremality shift are closely related (as we will see in detail later)

Corrections to extremality

$$\Delta\zeta = \frac{c}{201600\Lambda^6 m^6 \nu^{13}(1+\nu^2)^5} \times \Big[3239775\nu^{29} + 13046250\nu^{27} + 21354690\nu^{25} + 21664770\nu^{23} + 19661192\nu^{21}$$
$$+ 14479886\nu^{19} + 13943647\nu^{17} + 5093180\nu^{15} + 8429455\nu^{13} + 20545166\nu^{11}$$
$$+ 24409088\nu^{9} + 11319666\nu^{7} + 7270410\nu^{5} + 8419530\nu^{3} + 3239775\nu$$
$$+ 315(1+\nu^2)^5 \arctan\nu \times (10285\nu^{20} - 6580\nu^{18} + 10734\nu^{16} - 2548\nu^{14}$$
$$+ 709\nu^{12} - 10285\nu^{8} + 21268\nu^{6} - 34566\nu^{4} + 21268\nu^{2} - 10285)\Big]$$
$$+ \frac{\widetilde{c}}{22400\Lambda^6 m^6 \nu^{13}(1+\nu^2)^5} \times \Big[225225\nu^{29} + 1183350\nu^{27} + 2582790\nu^{25} + 3052830\nu^{23} + 2193344\nu^{21}$$
$$+ 1104562\nu^{19} + 518289\nu^{17} + 1336220\nu^{15} + 1603745\nu^{13} + 1589258\nu^{11}$$
$$+ 1577984\nu^{9} + 1619958\nu^{7} + 1505910\nu^{5} + 898590\nu^{3} + 225225\nu$$
$$+ 315(1+\nu^2)^5 \arctan\nu \times (715\nu^{20} + 420\nu^{18} + 138\nu^{16} + 84\nu^{14}$$
$$+ 43\nu^{12} - 715\nu^{8} + 484\nu^{6} - 938\nu^{4} + 484\nu^{2} - 715)\Big]$$

Corrections to extremality



$$\Delta\zeta > 0 \text{ for all } \nu$$

Terms of indefinite sign

Lower-order terms

- What about terms lower order in derivatives?

- For concreteness, and connection to the question of four-point amplitudes, we'll consider four-point operators among the graviton and photon:

$$F^4, \qquad R^2 F^2, \qquad R^4$$

Forward amplitudes:
$$\sim p^4, \qquad \sim p^6, \qquad \sim p^8$$
$$\sim s^2, \qquad \sim stu \to 0, \qquad \sim s^4$$

can bound   cannot bound   can bound

- If F^4 operators dominate, they give $\Delta z > 0$ Cheung, Liu, GR [1801.08546, 1903.09156]

- If R^4 operators dominate, they give $\Delta z > 0$

- Intermediate regime: Can $R^2 F^2$ operators of indefinite sign ever spoil positivity of Δz?

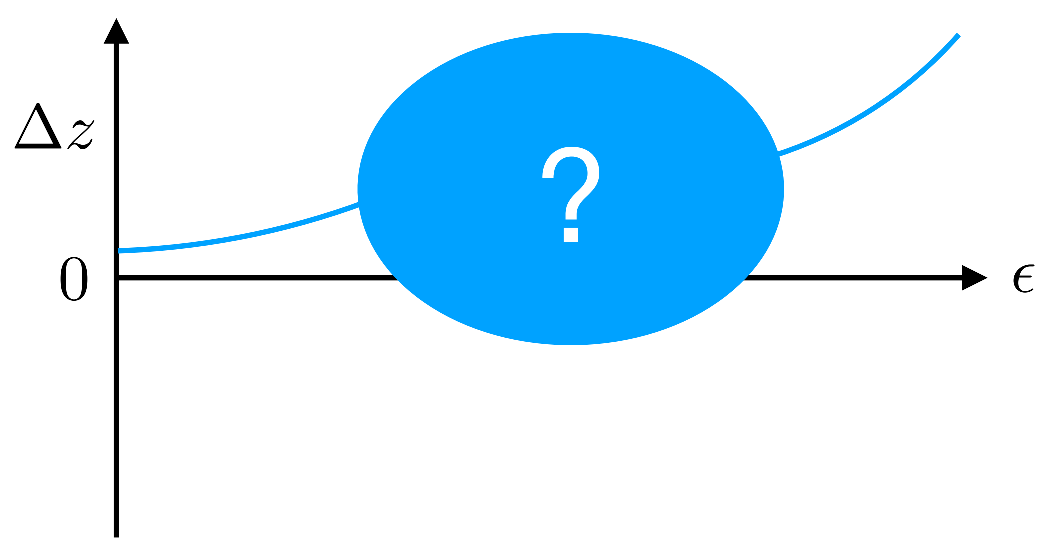
R^4, R^2F^2, F^4

- Example completion: massive scalar coupled to R^2 and F^2

$$\begin{aligned}\mathcal{L}_{\rm UV} =& \frac{R}{2\kappa^2} - \frac{1}{4}F^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m_\phi^2\phi^2 \\ &+ g_\phi \kappa \phi \left(a_1 R^2 + a_2 R_{ab}R^{ab} + a_3 R_{abcd}R^{abcd} + \epsilon b F^2\right)\end{aligned}$$

- Integrate out ϕ:

$$\Delta\mathcal{L} = \frac{\kappa^2 g_\phi^2}{2m_\phi^2}\left(a_1 R^2 + a_2 R_{ab}R^{ab} + a_3 R_{abcd}R^{abcd} + b\epsilon F^2\right)^2$$



Correction to extremality

- Can compute Δz either by solving the higher-derivative Einstein-Maxwell equations or using the on-shell action-integration method. Results agree.

- We find:

$$\Delta z = \frac{g_\phi^2}{45045 m^6} \times \big[252\kappa^4(55a_2^2 + 240a_2a_3 + 276a_3^2) \\ -5720(7a_2 + 17a_3)b\epsilon m^2\kappa^2\mu + 36036b^2\epsilon^2 m^4\mu^2\big]$$

Correction to extremality

- Can compute Δz either by solving the higher-derivative Einstein-Maxwell equations or using the on-shell action-integration method. Results agree.

- We find:

$$\Delta z = \frac{\kappa^4 g_\phi^2}{m^6 m_\phi^2} \times \left[\frac{46656}{614185} \left(\frac{95}{108} a_2 + a_3 \right)^2 + \frac{2352}{9449} \left(a_2 + \frac{17}{7} a_3 - \frac{9449}{5292} \frac{b\epsilon m^2}{\kappa^2} \mu \right)^2 + \frac{383}{59535} \frac{b^2 \epsilon^2 m^4}{\kappa^4} \mu^2 \right]$$

- Manifestly positive for black holes of arbitrary μ (electric, dyonic, magnetic)

BTZ black holes and the WGC

Arkani-Hamed, GR [19XX.XXXXX]

BTZ black holes

In 2+1 dimensions, let's consider charged BTZ black holes:

- Action:

$$\overline{\mathcal{L}} = \frac{1}{2\kappa^2}\left(R + \frac{2}{L^2}\right) - \frac{1}{4}F^2$$

- Charged black holes in the presence of a CC $\Lambda = -\frac{1}{L^2}$

- Metric:

$$\mathrm{d}s^2 = -\overline{g}(r)\mathrm{d}t^2 + \mathrm{d}r^2/\overline{g}(r) + r^2\mathrm{d}\phi^2$$

  where

$$\overline{g}(r) = -m + \tfrac{r^2}{L^2} - \tfrac{1}{2}q^2\log(r/r_0)$$

- Gauge field $F^{tr} = q/\sqrt{2}\kappa r$, charge $4\sqrt{2}\pi q/\kappa$

- Mass $\pi m/\kappa^2$ interior to reference radius r_0

BTZ black holes

In 2+1 dimensions, let's consider charged BTZ black holes:

- Extremality: For fixed q, m is bounded from below:

$$m \geq \frac{q^2}{4}\left[1 - \log\left(\frac{q^2 L^2}{4r_0^2}\right)\right] = m_{\min}(q)$$

  (cf. $m \geq q$ when $D \geq 4$)

- When $m = m_{\min}(q)$, the two horizons coincide at $\bar{r}_{\mathrm{H}} = Lq/2$.
  We will take the reference radius to satisfy $r_0 > Lq/2$.

- Extremality parameter: $z = m_{\min}(q)/m \in [0, 1]$

- In 3D, no propagating gravitons $\Longrightarrow$ no long-range gravitational force
  $\Longrightarrow$ usual formulation of WGC does not apply

3D Einstein-Maxwell EFT

- In $D = 3$ spacetime dimensions, $C_{\mu\nu\rho\sigma} = 0$ identically
  $\Longrightarrow$ can field-redefine all appearances of $R_{\mu\nu\rho\sigma}$ as polynomials in $F_{\mu\nu}$

- In 3D, all traces, e.g., $F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu}$, can be written entirely as powers of the Lorentz scalar $F^2 = F_{\mu\nu}F^{\mu\nu}$

- Let us take as our EFT action:

$$\mathcal{L} = \overline{\mathcal{L}} + \Delta\mathcal{L}$$
$$\Delta\mathcal{L} = c_n (F^2)^n$$

  for some $n \geq 2$

- 3D generalization of the Euler-Heisenberg action

Corrected BTZ solution

- Solving the nonlinearly perturbed Einstein-Maxwell equations, the metric is:

$$\mathrm{d}s^2 = -g(r)\mathrm{d}t^2 + \mathrm{d}r^2/g(r) + r^2\mathrm{d}\phi^2$$

$$g(r) = \overline{g}(r) + \frac{(-1)^n c_n q^{2n}}{(n-1)(\kappa r_0)^{2(n-1)}}\left[1 - \left(\frac{r_0}{r}\right)^{2(n-1)}\right]$$

- Correction to extremality parameter: physical black holes (with no naked singularity) satisfy $z \in [0, 1+\Delta z]$, where

$$\boxed{\Delta z = \frac{(-1)^n c_n q^2 (2/\kappa L)^{2(n-1)}}{(n-1)m_{\min}(q)}\left[1 - \left(\frac{qL}{2r_0}\right)^{2(n-1)}\right]}$$

$$\Delta z > 0 \iff (-1)^n c_n > 0$$

Dualization and scalar EFT

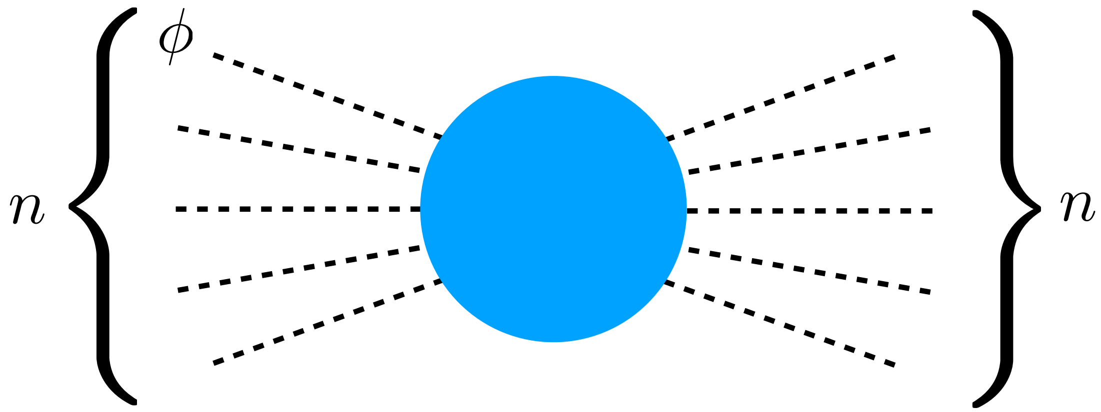
- To find a 3D version of the WGC for BTZ black holes, we need to bound c_n
- In 3D, can simplify the action by dualizing the photon to a scalar:

$$F_{\mu\nu} \to i\epsilon_{\mu\nu\rho}\partial^\rho\phi$$

- Gauge invariance has become shift symmetry of scalar
- We have $F^2 \to 2(\partial\phi)^2$ and $\Delta\mathcal{L} = 2^n c_n (\partial\phi)^{2n}$
- Example of a P(X) theory (where $X = (\partial\phi)^2$)
- To bound c_n from analyticity, we must consider n-to-n scattering of ϕs

Bounds on P(X) theory

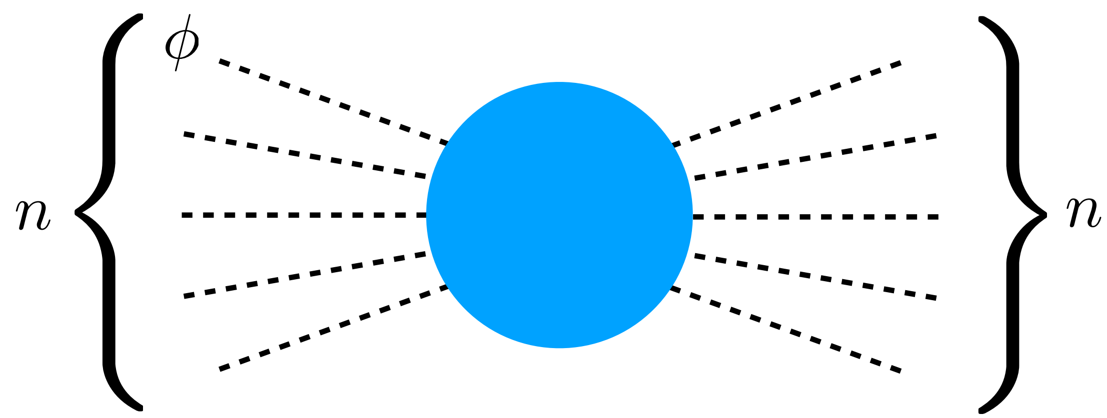
- To bound c_n from analyticity, we must consider n-to-n scattering of ϕs



Chandrasekaran, GR, Shahbazi-Moghaddam [1804.03153]

Bounds on P(X) theory

- To bound c_n from analyticity, we must consider n-to-n scattering of ϕs



- For even n, set:

$$\begin{aligned} p_i &= p_1, & i \text{ odd} \\ p_i &= p_2, & i \text{ even} \\ p_{i+n} &= -p_i \\ i &\in \{1, \ldots, n\} \end{aligned} \qquad \Longrightarrow \qquad \mathcal{A}(s) = (n!)^2 \left(\frac{2}{n}\right)^{2n} c_n s^n$$

and continue in $s = \frac{n^2}{4} s_{12}$

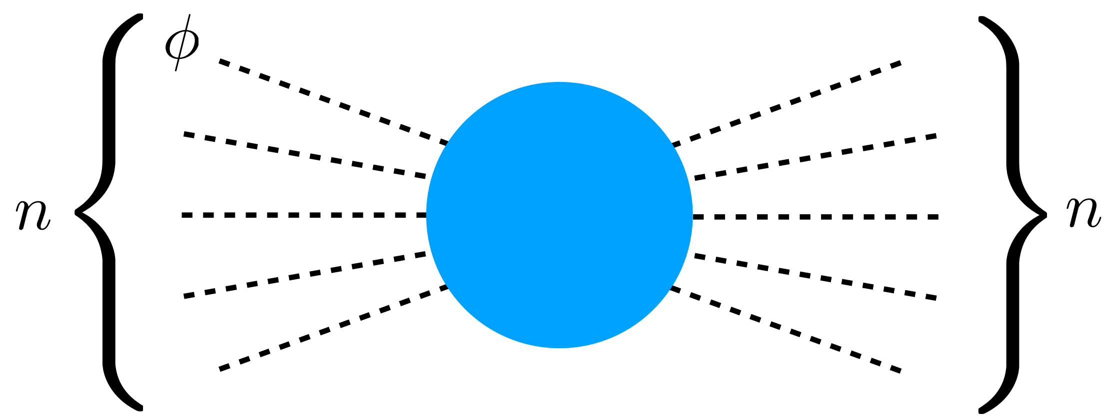
- Dispersion relation gives:

$$c_n = \frac{1}{\pi (n!)^2} \left(\frac{n}{2}\right)^{2n} \int_0^\infty \frac{\mathrm{d}s}{s^{n+1}} \sum_X \int \mathrm{dLIPS}_X |\mathcal{A}(|n, s\rangle \to X)|^2 > 0$$

Chandrasekaran, GR, Shahbazi-Moghaddam [1804.03153]

Bounds on P(X) theory

- To bound c_n from analyticity, we must consider n-to-n scattering of ϕs



- For odd n, set:

$$
\begin{aligned}
p_i &= p_1, && i \text{ odd}, \ \ i \in \{1, \ldots, n-1\} \\
p_i &= p_2, && i \text{ even}, \ \ i \in \{1, \ldots, n-1\} \\
p_{i+n} &= -p_i, && i \in \{1, \ldots, n\}
\end{aligned}
$$

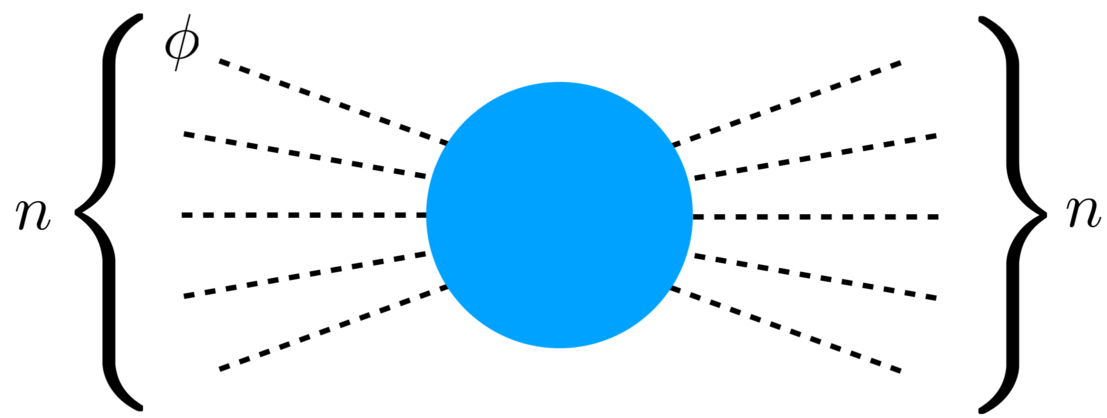
$$
\implies \quad \mathcal{A}(s) = -\left(\frac{2}{n-1}\right)^{2n-3} n!(n-1)!\delta^{n-2} c_n (s-\delta)^2
$$

and continue in $s = (n-1)s_{1n} + \delta, \ \ \delta = (n-1)^2 s_{12}/4$ fixed

Chandrasekaran, GR, Shahbazi-Moghaddam [1804.03153]

Bounds on P(X) theory

- To bound c_n from analyticity, we must consider n-to-n scattering of ϕs



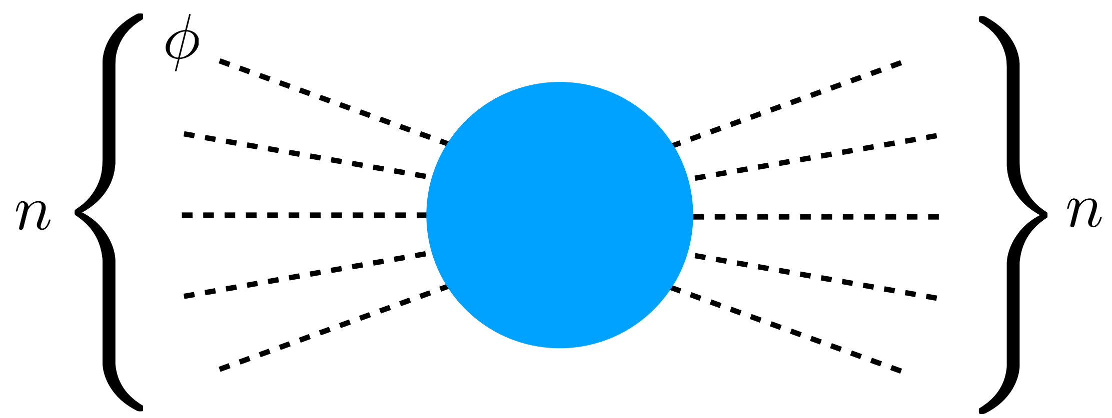
- For odd n, dispersion relation gives:

$$c_n = -\frac{1}{2\pi\, n!(n-1)!\delta^{n-2}}\left(\frac{n-1}{2}\right)^{2n-3} \times$$
$$\times \int_0^\infty \frac{\mathrm{d}s}{s^{n+1}} \sum_X \int \mathrm{dLIPS}_X \left[|\mathcal{A}(|n,s\rangle \to X)|^2 + |\mathcal{A}(|n,s+2\delta\rangle \to X)|^2\right] < 0$$

Chandrasekaran, GR, Shahbazi-Moghaddam [1804.03153]

Bounds on P(X) theory

- To bound c_n from analyticity, we must consider n-to-n scattering of ϕs



- As shown in Chandrasekaran, GR, Shahbazi-Moghaddam [1804.03153] , analyticity of the forward amplitude at complex momenta requires $(-1)^n c_n > 0$

- Generalization of well known positivity of $(\partial\phi)^4$ coefficient Adams et al. [hep-th/0602178]

$$\Longrightarrow \quad \Delta z > 0$$

Four-derivative, dyonic and spinning

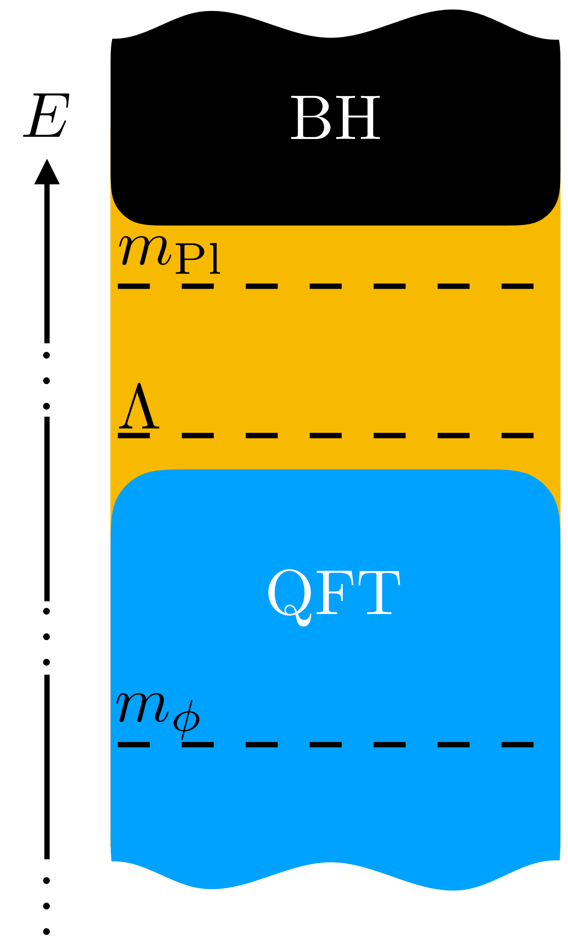
Cheung, Liu, GR [1903.09156, 1801.08546]

Assumptions

A final tool: self-consistency of black hole thermodynamics

For the purposes of this proof, we assume:

1. There exist quantum fields ϕ at a mass scale m_ϕ satisfying
$$m_\phi \ll \Lambda,$$
where Λ is the scale at which QFT breaks down.
In general, Λ can be much smaller than the Planck scale.



Assumptions

A final tool: self-consistency of black hole thermodynamics

For the purposes of this proof, we assume:

1. There exist quantum fields ϕ at a mass scale m_ϕ satisfying
   $$m_\phi \ll \Lambda,$$
   where Λ is the scale at which QFT breaks down.
   In general, Λ can be much smaller than the Planck scale.

2. The fields ϕ couple to photons and gravitons so that the higher-dimension operators are generated at tree level, e.g., $\sim \phi R,\ \phi F^2$
   so:
   $$\underbrace{c_i \propto 1/m_\phi^2}_{\text{QFT effect}} \gg \underbrace{1/\Lambda^2}_{\text{Quantum gravity “slop”}}$$
   Couplings like this are common in string theory: dilaton and moduli are massless in supersymmetric limit, and acquire masses if SUSY is broken.

Assumptions

A final tool: self-consistency of black hole thermodynamics

For the purposes of this proof, we assume:

1. There exist quantum fields ϕ at a mass scale m_ϕ satisfying
$$m_\phi \ll \Lambda,$$
where Λ is the scale at which QFT breaks down.
In general, Λ can be much smaller than the Planck scale.

2. The fields ϕ couple to photons and gravitons so that the higher-dimension operators are generated at tree level, e.g., $\sim \phi R,\ \phi F^2$
so:
$$\underbrace{c_i \propto 1/m_\phi^2}_{\text{QFT effect}} \gg \underbrace{1/\Lambda^2}_{\text{Quantum gravity “slop”}}$$
Couplings like this are common in string theory: dilaton and moduli are massless in supersymmetric limit, and acquire masses if SUSY is broken.

3. We will consider black holes with charge and spin chosen such that they are thermodynamically stable in the path integral.

Thermodynamic stability

Heat capacity at constant J:

$$C_J = \left(\frac{\partial M}{\partial T}\right)_J = T\left(\frac{\partial S}{\partial T}\right)_J$$

Temperature: $T = \dfrac{\xi}{2\pi m(1+\nu^2)(1+\xi)^2}$

Thermodynamic stability

Heat capacity at constant J:

$$C_J = \left(\frac{\partial M}{\partial T}\right)_J = T\left(\frac{\partial S}{\partial T}\right)_J$$

Temperature: $T = \dfrac{\xi}{2\pi m(1+\nu^2)(1+\xi)^2}$

Black hole's angular velocity:

$$\Omega = \frac{a}{a^2 + r_{\mathrm{H}}^2}$$

Isothermal moment of inertia:

$$\epsilon^{ij} = \left(\frac{\partial J^i}{\partial \Omega_j}\right)_T$$

Thermodynamic stability of partition function in grand canonical ensemble requires $C_J > 0$ and $\mathrm{spec}(\epsilon^{ij}) > 0$ Monteiro, Perry, Santos [0903.3256]

Thermodynamic stability

Heat capacity at constant J:

$$C_J = \left(\frac{\partial M}{\partial T}\right)_J = T\left(\frac{\partial S}{\partial T}\right)_J$$

Temperature: $T = \dfrac{\xi}{2\pi m(1+\nu^2)(1+\xi)^2}$

Black hole's angular velocity:

$$\Omega = \frac{a}{a^2 + r_{\mathrm{H}}^2}$$

Isothermal moment of inertia:

$$\epsilon^{ij} = \left(\frac{\partial J^i}{\partial \Omega_j}\right)_T$$

For our KN black hole, this becomes the requirement:

$$\boxed{0 < \xi < \frac{1-3\nu^2}{2(1+\nu^2)}}$$

When $\nu = 0$, require $\xi \in (0, \frac{1}{2})$, i.e., $q^2 + \tilde{q}^2 > \frac{3}{4}m^2$

When $\xi = 0$, require $\nu \in [0, 1/\sqrt{3})$, i.e., $q^2 + \tilde{q}^2 > 2a^2$

Euclidean path integral

We want to compute ΔS, the shift in Wald entropy for a black hole with fixed M, Q, J (as measured at infinity), in theory $\mathcal{L}$ vs. $\overline{\mathcal{L}}$

Euclidean path integral

We want to compute ΔS, the shift in Wald entropy for a black hole with fixed M, Q, J (as measured at infinity), in theory $\mathcal{L}$ vs. $\overline{\mathcal{L}}$

For now, work in grand canonical ensemble: fixed T, Ω

Inverse temperature β defines periodicity in Euclidean time for the Euclidean path integral,

$$\boxed{e^{-\beta G} = Z = \int d[\hat{g}]d[\hat{A}]e^{-I[\hat{g},\hat{A}]}}$$

where

$I = \overline{I} + \Delta I$ is the Euclidean action (spacetime integral of Wick-rotated Lagrangian)

$G = M - TS - \Omega J$ is the Gibbs free energy

$\hat{g}, \hat{A}$ are integration variables for the metric and gauge field

Gibbons, Hawking (1976)

Euclidean path integral

We want to compute ΔS, the shift in Wald entropy for a black hole with fixed M, Q, J (as measured at infinity), in theory $\mathcal{L}$ vs. $\overline{\mathcal{L}}$

For now, work in grand canonical ensemble: fixed T, Ω

Ultraviolet completion: introduce integration variable ϕ for the heavy fields that are integrated out when we go from UV to IR:

$$\int d[\hat{g}]d[\hat{A}]d[\hat{\phi}]\, e^{-I_{\rm UV}[\hat{g},\hat{A},\hat{\phi}]} = \int d[\hat{g}]d[\hat{A}]\, e^{-I[\hat{g},\hat{A}]}$$

We define the vev of $\hat{\phi}$ to be zero in flat space.

For the on-shell black hole in the $\mathcal{L}$ theory, $\phi \neq 0$, since equations of motion dictate $\phi \sim R, F^2$

Going off shell

We can evaluate the Euclidean action at any field configuration we wish, including one that does *not* satisfy the classical equations of motion.

In particular, let's evaluate I_{UV} at $\hat{\phi} = 0$, which turns off all the higher-dimension operators in $\Delta\mathcal{L}$, so we have the simple mathematical fact:

$$\boxed{I_{\text{UV}}[\hat{g}, \hat{A}, 0] = \overline{I}[\hat{g}, \hat{A}]}$$

where $\overline{I}$ is the Euclidean action for pure Einstein-Maxwell theory.

This observation will allow us to compare the two black hole entropies in $\mathcal{L}$ and $\overline{\mathcal{L}}$ via an argument that only involves working in a *single* theory.

Euclidean action identity

Following Reall, Santos [1901.11535], we have:

$$\Delta(\beta G)|_{T,\Omega} = \beta\Delta M|_{T,\Omega} - \beta\Omega\Delta J|_{T,\Omega} - \Delta S|_{T,\Omega}$$

by the definition of the Gibbs free energy. Throughout, Δ denotes shifts to first order in the c_i.

Euclidean action identity

Following Reall, Santos [1901.11535], we have:

$$\Delta(\beta G)|_{T,\Omega} = \beta \Delta M|_{T,\Omega} - \beta\Omega\Delta J|_{T,\Omega} - \Delta S|_{T,\Omega}$$

But we can reparameterize:

$$\Delta S|_{T,\Omega} = \left(\frac{\partial S}{\partial M}\right)_J \Delta M|_{T,\Omega} + \left(\frac{\partial S}{\partial J}\right)_M \Delta J|_{T,\Omega} + \Delta S|_{M,J}$$

Euclidean action identity

Following Reall, Santos [1901.11535], we have:

$$\Delta(\beta G)|_{T,\Omega} = \beta \Delta M|_{T,\Omega} - \beta\Omega\Delta J|_{T,\Omega} - \Delta S|_{T,\Omega}$$

But we can reparameterize:

$$\Delta S|_{T,\Omega} = \underset{\substack{\uparrow \\ = \beta}}{\left(\frac{\partial S}{\partial M}\right)_J} \Delta M|_{T,\Omega} + \underset{\substack{\uparrow \\ = -\beta\Omega}}{\left(\frac{\partial S}{\partial J}\right)_M} \Delta J|_{T,\Omega} + \Delta S|_{M,J}$$

by first law $dM = TdS + \Omega dJ$

Euclidean action identity

Following Reall, Santos [1901.11535], we have:

$$\Delta(\beta G)|_{T,\Omega} = \beta \Delta M|_{T,\Omega} - \beta\Omega\Delta J|_{T,\Omega} - \Delta S|_{T,\Omega}$$

But we can reparameterize:

$$\Delta S|_{T,\Omega} = \left(\frac{\partial S}{\partial M}\right)_J \Delta M|_{T,\Omega} + \left(\frac{\partial S}{\partial J}\right)_M \Delta J|_{T,\Omega} + \Delta S|_{M,J}$$

$$\implies \Delta(\beta G)|_{T,\Omega} = -\Delta S|_{M,J}$$

Euclidean action identity

Following Reall, Santos [1901.11535], we have:

$$\Delta(\beta G)|_{T,\Omega} = \beta\Delta M|_{T,\Omega} - \beta\Omega\Delta J|_{T,\Omega} - \Delta S|_{T,\Omega}$$

But we can reparameterize:

$$\Delta S|_{T,\Omega} = \left(\frac{\partial S}{\partial M}\right)_J \Delta M|_{T,\Omega} + \left(\frac{\partial S}{\partial J}\right)_M \Delta J|_{T,\Omega} + \Delta S|_{M,J}$$

$$\implies \Delta(\beta G)|_{T,\Omega} = -\Delta S|_{M,J}$$

Smarr relation $\beta G = I$ then implies:

$$\boxed{\Delta S = -\Delta I}$$

where right-hand side is evaluated on the Kerr-Newman solution.

Free energy inequality

Putting our thermodynamic argument together, we have the string of (in)equalities relating the Euclidean actions of an Einstein-Maxwell and perturbed Kerr-Newman black hole:

$$\Delta I = I[g_{M,J,Q}, A_{M,J,Q}] - \overline{I}[\overline{g}_{M,J,Q}, \overline{A}_{M,J,Q}]$$

by definition of ΔI, where $g_{M,J,Q}, A_{M,J,Q}$ and $\overline{g}_{M,J,Q}, \overline{A}_{M,J,Q}$, respectively, are solutions to the perturbed and unperturbed Einstein-Maxwell equations at fixed M, J, Q

Free energy inequality

Putting our thermodynamic argument together, we have the string of (in)equalities relating the Euclidean actions of an Einstein-Maxwell and perturbed Kerr-Newman black hole:

$$\begin{aligned}\Delta I &= I[g_{M,J,Q}, A_{M,J,Q}] - \overline{I}[\overline{g}_{M,J,Q}, \overline{A}_{M,J,Q}] \\ &= I_{\mathrm{UV}}[g_{M,J,Q}, A_{M,J,Q}, \phi_{M,J,Q}] - \overline{I}[\overline{g}_{M,J,Q}, \overline{A}_{M,J,Q}]\end{aligned}$$

by the saddle-point approximation

Free energy inequality

Putting our thermodynamic argument together, we have the string of (in)equalities relating the Euclidean actions of an Einstein-Maxwell and perturbed Kerr-Newman black hole:

$$\begin{aligned}\Delta I &= I[g_{M,J,Q}, A_{M,J,Q}] - \overline{I}[\overline{g}_{M,J,Q}, \overline{A}_{M,J,Q}] \\ &= I_{\rm UV}[g_{M,J,Q}, A_{M,J,Q}, \phi_{M,J,Q}] - \overline{I}[\overline{g}_{M,J,Q}, \overline{A}_{M,J,Q}] \\ &= I_{\rm UV}[g_{M,J,Q}, A_{M,J,Q}, \phi_{M,J,Q}] - I_{\rm UV}[\overline{g}_{M,J,Q}, \overline{A}_{M,J,Q}, 0]\end{aligned}$$

by the off-shell relation we found previously, relating $I_{\rm UV}$ and $\overline{I}$

Free energy inequality

Putting our thermodynamic argument together, we have the string of (in)equalities relating the Euclidean actions of an Einstein-Maxwell and perturbed Kerr-Newman black hole:

$$\begin{aligned}
\Delta I &= I[g_{M,J,Q}, A_{M,J,Q}] - \overline{I}[\overline{g}_{M,J,Q}, \overline{A}_{M,J,Q}] \\
&= I_{\mathrm{UV}}[g_{M,J,Q}, A_{M,J,Q}, \phi_{M,J,Q}] - \overline{I}[\overline{g}_{M,J,Q}, \overline{A}_{M,J,Q}] \\
&= I_{\mathrm{UV}}[g_{M,J,Q}, A_{M,J,Q}, \phi_{M,J,Q}] - I_{\mathrm{UV}}[\overline{g}_{M,J,Q}, \overline{A}_{M,J,Q}, 0] \\
&= I_{\mathrm{UV}}[g_{M,J,Q}, A_{M,J,Q}, \phi_{M,J,Q}] - I_{\mathrm{UV}}[g_{M,J,Q}, A_{M,J,Q}, 0]
\end{aligned}$$

since we wish to compute ΔI to first order in the c_i

Free energy inequality

Putting our thermodynamic argument together, we have the string of (in)equalities relating the Euclidean actions of an Einstein-Maxwell and perturbed Kerr-Newman black hole:

$$\begin{aligned}
\Delta I &= I[g_{M,J,Q}, A_{M,J,Q}] - \overline{I}[\overline{g}_{M,J,Q}, \overline{A}_{M,J,Q}] \\
&= I_{\mathrm{UV}}[g_{M,J,Q}, A_{M,J,Q}, \phi_{M,J,Q}] - \overline{I}[\overline{g}_{M,J,Q}, \overline{A}_{M,J,Q}] \\
&= I_{\mathrm{UV}}[g_{M,J,Q}, A_{M,J,Q}, \phi_{M,J,Q}] - I_{\mathrm{UV}}[\overline{g}_{M,J,Q}, \overline{A}_{M,J,Q}, 0] \\
&= I_{\mathrm{UV}}[g_{M,J,Q}, A_{M,J,Q}, \phi_{M,J,Q}] - I_{\mathrm{UV}}[g_{M,J,Q}, A_{M,J,Q}, 0] \\
&< 0
\end{aligned}$$

if the saddle-point of the solution is a local minimum
(will justify shortly)

Free energy inequality

Putting our thermodynamic argument together, we have the string of (in)equalities relating the Euclidean actions of an Einstein-Maxwell and perturbed Kerr-Newman black hole:

$$\begin{aligned}
\Delta I &= I[g_{M,J,Q}, A_{M,J,Q}] - \overline{I}[\overline{g}_{M,J,Q}, \overline{A}_{M,J,Q}] \\
&= I_{\mathrm{UV}}[g_{M,J,Q}, A_{M,J,Q}, \phi_{M,J,Q}] - \overline{I}[\overline{g}_{M,J,Q}, \overline{A}_{M,J,Q}] \\
&= I_{\mathrm{UV}}[g_{M,J,Q}, A_{M,J,Q}, \phi_{M,J,Q}] - I_{\mathrm{UV}}[\overline{g}_{M,J,Q}, \overline{A}_{M,J,Q}, 0] \\
&= I_{\mathrm{UV}}[g_{M,J,Q}, A_{M,J,Q}, \phi_{M,J,Q}] - I_{\mathrm{UV}}[g_{M,J,Q}, A_{M,J,Q}, 0] \\
&< 0
\end{aligned}$$

$$\Longrightarrow \quad \boxed{\Delta S > 0}$$

Free energy inequality

Putting our thermodynamic argument together, we have the string of (in)equalities relating the Euclidean actions of an Einstein-Maxwell and perturbed Kerr-Newman black hole:

$$
\begin{aligned}
\Delta I &= I[g_{M,J,Q}, A_{M,J,Q}] - \overline{I}[\overline{g}_{M,J,Q}, \overline{A}_{M,J,Q}] \\
&= I_{\mathrm{UV}}[g_{M,J,Q}, A_{M,J,Q}, \phi_{M,J,Q}] - \overline{I}[\overline{g}_{M,J,Q}, \overline{A}_{M,J,Q}] \\
&= I_{\mathrm{UV}}[g_{M,J,Q}, A_{M,J,Q}, \phi_{M,J,Q}] - I_{\mathrm{UV}}[\overline{g}_{M,J,Q}, \overline{A}_{M,J,Q}, 0] \\
&= I_{\mathrm{UV}}[g_{M,J,Q}, A_{M,J,Q}, \phi_{M,J,Q}] - I_{\mathrm{UV}}[g_{M,J,Q}, A_{M,J,Q}, 0] \\
&< 0
\end{aligned}
$$

$$\Longrightarrow \quad \boxed{\Delta S > 0}$$

Accords with our intuition that integrating out extra massive QFT degrees of freedom should increase the black hole’s entropy. Cheung, Liu, GNR [1801.08546]

Free energy inequality

Putting our thermodynamic argument together, we have the string of (in)equalities relating the Euclidean actions of an Einstein-Maxwell and perturbed Kerr-Newman black hole:

$$\begin{aligned}\Delta I &= I[g_{M,J,Q}, A_{M,J,Q}] - \overline{I}[\overline{g}_{M,J,Q}, \overline{A}_{M,J,Q}] \\ &= I_{\rm UV}[g_{M,J,Q}, A_{M,J,Q}, \phi_{M,J,Q}] - \overline{I}[\overline{g}_{M,J,Q}, \overline{A}_{M,J,Q}] \\ &= I_{\rm UV}[g_{M,J,Q}, A_{M,J,Q}, \phi_{M,J,Q}] - I_{\rm UV}[\overline{g}_{M,J,Q}, \overline{A}_{M,J,Q}, 0] \\ &= I_{\rm UV}[g_{M,J,Q}, A_{M,J,Q}, \phi_{M,J,Q}] - I_{\rm UV}[g_{M,J,Q}, A_{M,J,Q}, 0] \\ &< 0\end{aligned}$$

$$\implies \boxed{\Delta S > 0}$$

Since ∂_t is a Killing vector for Kerr-Newman, the on-shell Euclidean action is $\Delta I = -\beta \int \mathrm{d}^3x \sqrt{-g}\Delta\mathcal{L}|_{\rm KN}$, where the right-hand side is evaluated on the Lorentzian Kerr-Newman solution. Reall, Santos [1901.11535]

Free energy inequality

Putting our thermodynamic argument together, we have the string of (in)equalities relating the Euclidean actions of an Einstein-Maxwell and perturbed Kerr-Newman black hole:

$$
\begin{aligned}
\Delta I &= I[g_{M,J,Q}, A_{M,J,Q}] - \overline{I}[\overline{g}_{M,J,Q}, \overline{A}_{M,J,Q}] \\
&= I_{\mathrm{UV}}[g_{M,J,Q}, A_{M,J,Q}, \phi_{M,J,Q}] - \overline{I}[\overline{g}_{M,J,Q}, \overline{A}_{M,J,Q}] \\
&= I_{\mathrm{UV}}[g_{M,J,Q}, A_{M,J,Q}, \phi_{M,J,Q}] - I_{\mathrm{UV}}[\overline{g}_{M,J,Q}, \overline{A}_{M,J,Q}, 0] \\
&= I_{\mathrm{UV}}[g_{M,J,Q}, A_{M,J,Q}, \phi_{M,J,Q}] - I_{\mathrm{UV}}[g_{M,J,Q}, A_{M,J,Q}, 0] \\
&< 0
\end{aligned}
$$

$$\implies \quad \boxed{\Delta S > 0}$$

We have:

$$\boxed{\Delta S = \beta \int \mathrm{d}^3x \sqrt{-g}\Delta\mathcal{L}|_{\mathrm{KN}}}$$

Minimization of the Euclidean action

- We needed the saddle point, corresponding to the classical solution, to be a local minimum. Equivalently, we needed the Euclidean action to be stable under small off-shell perturbations.

- What about conformal saddle-point instabilities? These have been shown to be gauge artifacts: one can decompose the metric and integrate over these unphysical instabilities separately in the path integral. Gibbons, Hawking, Perry (1978); Gibbons, Perry (1978)

- The Euclidean Schwarzschild black hole is known to have a bona fide instability. Gross, Perry, Yaffe (1982)

- However, for Kerr-Newman black holes, this instability is always connected with thermodynamic instabilities in the partition function (negative specific heat or negative isothermal moment of inertia)
  Prestidge [hep-th/9907163]; Reall [hep-th/0104071]; Monteiro, Santos [0812.1767]; Monteiro, Perry, Santos [0909.3256]

- Hence, we enforce our thermodynamic stability condition, $0 < \xi < \frac{1-3\nu^2}{2(1+\nu^2)}$, which will ensure that our $\Delta S > 0$ argument is valid.

Four-derivative EM effective action

Let's return to the four-derivative corrections to Einstein-Maxwell theory:

$$\begin{aligned}\Delta\mathcal{L} = \quad & c_1 R^2 + c_2 R_{\mu\nu} R^{\mu\nu} + c_3 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \\ & + c_4 R F_{\mu\nu} F^{\mu\nu} + c_5 R_{\mu\nu} F^{\mu\rho} F^{\nu}{}_{\rho} + c_6 R_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} \\ & + c_7 F_{\mu\nu} F^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} + c_8 F_{\mu\nu} F^{\nu\rho} F_{\rho\sigma} F^{\sigma\mu}\end{aligned}$$

We will then use the $\Delta S > 0$ condition to obtain a family a positivity bounds on combinations of the c_i.

The electric, nonspinning case was considered in Cheung, Liu, GR [1801.08546] .

We will generalize to black holes that are spinning and dyonically-charged, parameterized by $M, Q, \widetilde{Q}, J$, or equivalently, m, μ, ν, ξ.

Define: $d_{1,2,3} = \kappa^2 c_{1,2,3}, \qquad d_{4,5,6} = c_{4,5,6}, \qquad d_{7,8} = \kappa^{-2} c_{7,8}$

Field redefinition invariance

Consider a general field redefinition of the metric:

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu} = g_{\mu\nu} + r_1 R_{\mu\nu} + r_2 g_{\mu\nu} R + r_3 \kappa^2 F_{\mu\rho} F_\nu{}^\rho + r_4 \kappa^2 g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}$$

This has the effect of shifting the action, $\delta\mathcal{L} = \frac{1}{2\kappa^2} \delta g^{\mu\nu} \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \kappa^2 T_{\mu\nu} \right)$

which has the net effect of shifting the higher-dimension operator coefficients:

$$d_1 \to d_1 - \frac{1}{4} r_1 - \frac{1}{2} r_2 \qquad d_5 \to d_5 - \frac{1}{2} r_1 + \frac{1}{2} r_3$$

$$d_2 \to d_2 + \frac{1}{2} r_1 \qquad d_6 \to d_6$$

$$d_3 \to d_3 \qquad d_7 \to d_7 + \frac{1}{8} r_3$$

$$d_4 \to d_4 + \frac{1}{8} r_1 - \frac{1}{4} r_3 - \frac{1}{2} r_4 \qquad d_8 \to d_8 - \frac{1}{2} r_3$$

Field redefinition invariance

There are four combinations of higher-dimension operator coefficients that are invariant under this transformation:

$$d_0 = d_2 + 4d_3 + d_5 + d_6 + 4d_7 + 2d_8$$

$$d_3$$

$$d_6$$

$$d_9 = d_2 + 4d_3 + d_5 + 2d_6 + d_8$$

The entropy shift, $\Delta S = \beta \int \mathrm{d}^3x \sqrt{-g} \Delta\mathcal{L}|_{\mathrm{KN}}$, will automatically be field-redefinition invariant, since the transformation is proportional to the Einstein equations. Hence, it is built out of $d_0,\ d_3,\ d_6,\ d_9$.

EFT bounds

Cheung, Liu, GR [1903.09156]

EFT bounds

Rotating dyonic black hole entropy shift:

$$\boxed{\begin{aligned}\Delta S(\xi,\mu,\nu) = &\\ &\frac{16\pi^2(2\mu^2-1)(\nu^2-3)(3\nu^2-1)[1-\xi-\nu^2(1+\xi)]^2}{15\kappa^2\xi(1+\xi)(1+\nu^2)^5}(d_0+d_6-d_9)\\ &+\frac{\pi^2[1-\xi-\nu^2(1+\xi)]^2[\nu(3+2\nu^2+3\nu^4)+3(\nu^2-1)(1+\nu^2)^2\arctan\nu]}{2\kappa^2\xi(1+\xi)\nu^5(1+\nu^2)}(d_0+d_6+d_9)\\ &+\frac{64\pi^2}{\kappa^2(1+\nu^2)}d_3+\frac{32\pi^2\mu[1-\xi-\nu^2(1+\xi)][\nu^2(3+4\xi)-1-4\xi]}{5\kappa^2\xi(1+\xi)(1+\nu^2)^3}d_6.\end{aligned}}$$

Consistency of black hole entropy demands that

$$\boxed{\Delta S(\xi,\mu,\nu)>0}$$

for all

$$-1\leq\mu\leq 1 \quad \text{and} \quad 0<\xi<\frac{1-3\nu^2}{2(1+\nu^2)}$$

A 3-parameter family of consistency bounds on all corrections to Einstein-Maxwell theory generated by tree-level QFT

EFT bounds

Numerical tests indicate that the strongest bounds come from the nonspinning case. That is, the $\nu = 0$ case implies the full family of bounds:

$$\Delta S(\xi,\mu,0) = \frac{32\pi^2}{5\kappa^2\xi(1+\xi)} \times \Big\{(1-\xi)^2[\mu^2 d_0 + (1-\mu^2)d_9] + 10\xi(1+\xi)d_3 \\ + \mu(1-\xi)[\mu(1-\xi) - 1 - 4\xi]d_6\Big\} \\ > 0$$

for all $-1 \leq \mu \leq 1$ and $0 < \xi < \frac{1}{2}$

Two-parameter family of bounds, convex bound space.

Consequences of EFT bounds

Let us explore consequences of our family of EFT bounds:

Bounding $d_{0,9}$:

$$\Delta S(0,1,0) \implies d_0 > 0$$
$$\Delta S(0,0,0) \implies d_9 > 0$$

Consequences of EFT bounds

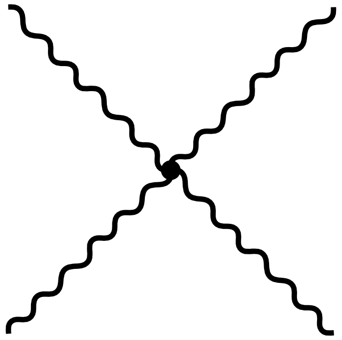
Let us explore consequences of our family of EFT bounds:

Bounding $d_{0,9}$:

If only F^4 terms are present, these bounds become

$$2d_7 + d_8 > 0$$
$$d_8 > 0$$

which follow from analyticity of four-photon scattering Adams et al. [hep-th/0602178]; Cheung, GR [1407.7865]

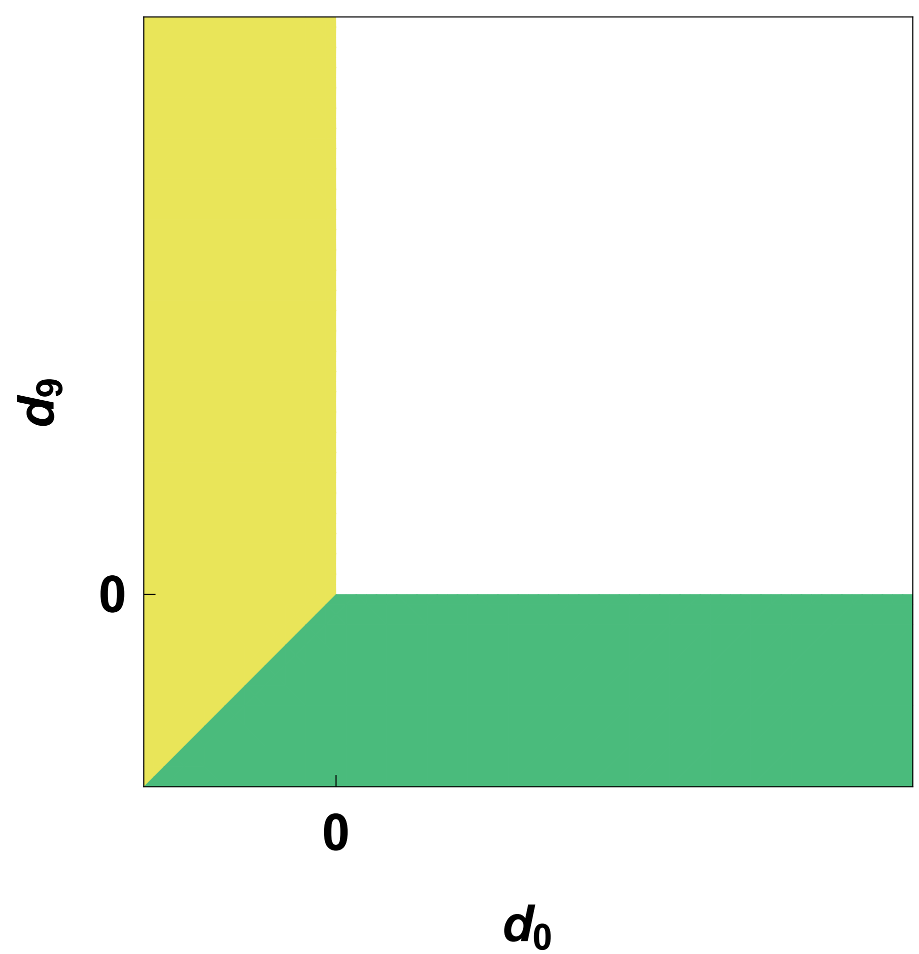


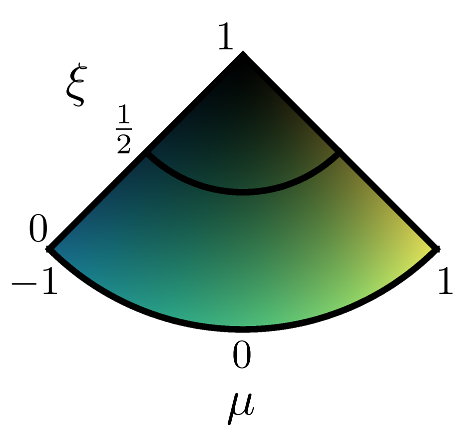
Consequences of EFT bounds

Let us explore consequences of our family of EFT bounds:

Bounding $d_{0,9}$:

Excluded regions:





Consequences of EFT bounds

Let us explore consequences of our family of EFT bounds:

Bounding d_3:

$$\begin{aligned} \Delta S(\tfrac{1}{2},0,0) > 0 &\implies d_3/d_9 > -1/30 \\ 7\Delta S(\tfrac{1}{2},1,0) + 5\Delta S(\tfrac{1}{2},-1,0) > 0 &\implies d_3/d_0 > -1/30 \end{aligned}$$

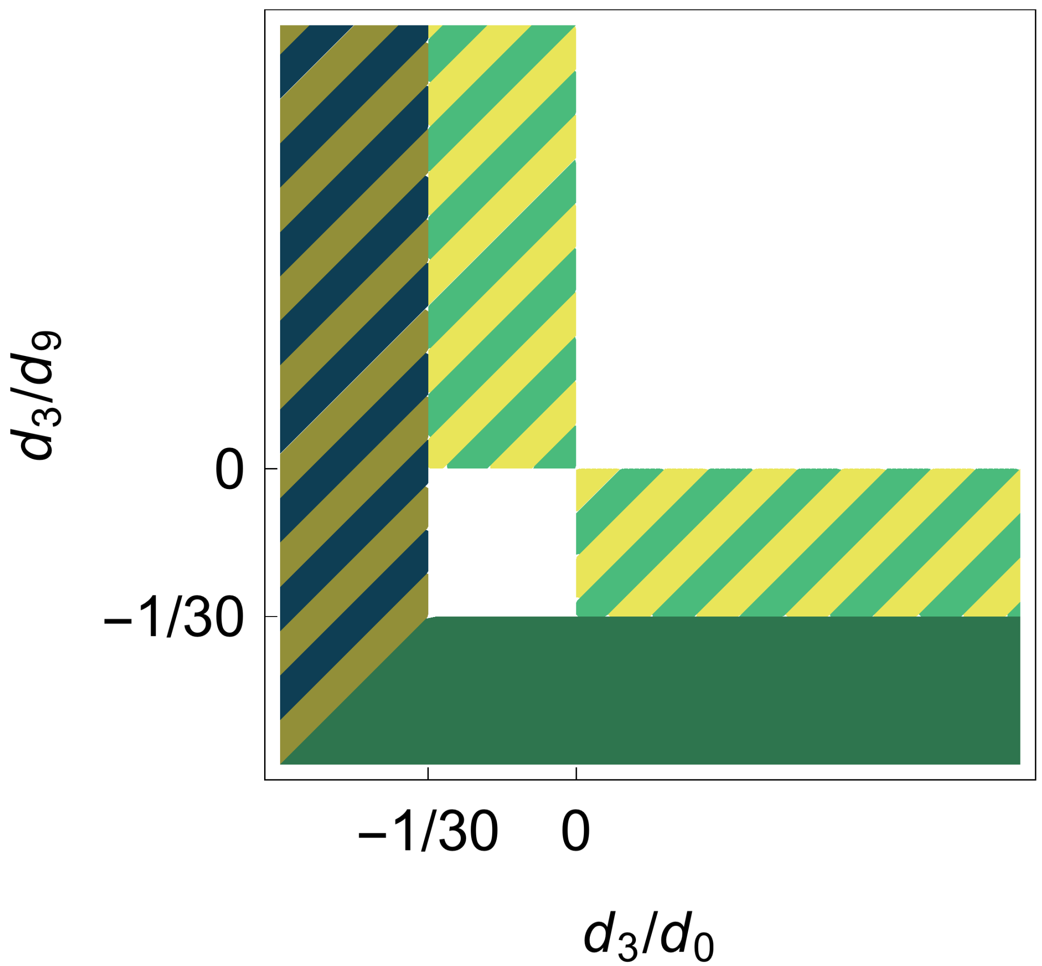
An argument for positivity of d_3 from unitarity was given in Cheung, GR [1608.02942]

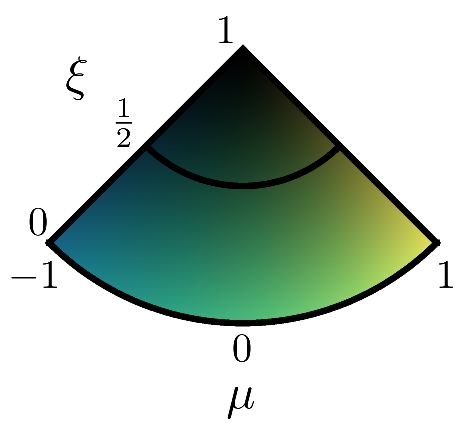
Consequences of EFT bounds

Let us explore consequences of our family of EFT bounds:

Bounding d_3:

Excluded regions:





Consequences of EFT bounds

Let us explore consequences of our family of EFT bounds:

Bounding d_6:

Marginalize over $\Delta S(0,\mu,0)$ to obtain the optimal bounds:

$$\left[\begin{array}{ll} 2(d_9-\sqrt{d_0 d_9}), & 4d_9 \leq d_0 \\ -d_0/2, & 4d_9 > d_0 \end{array}\right] < d_6 < 2(d_9+\sqrt{d_0 d_9})$$

For SUSY theories, $d_6 = 0$.

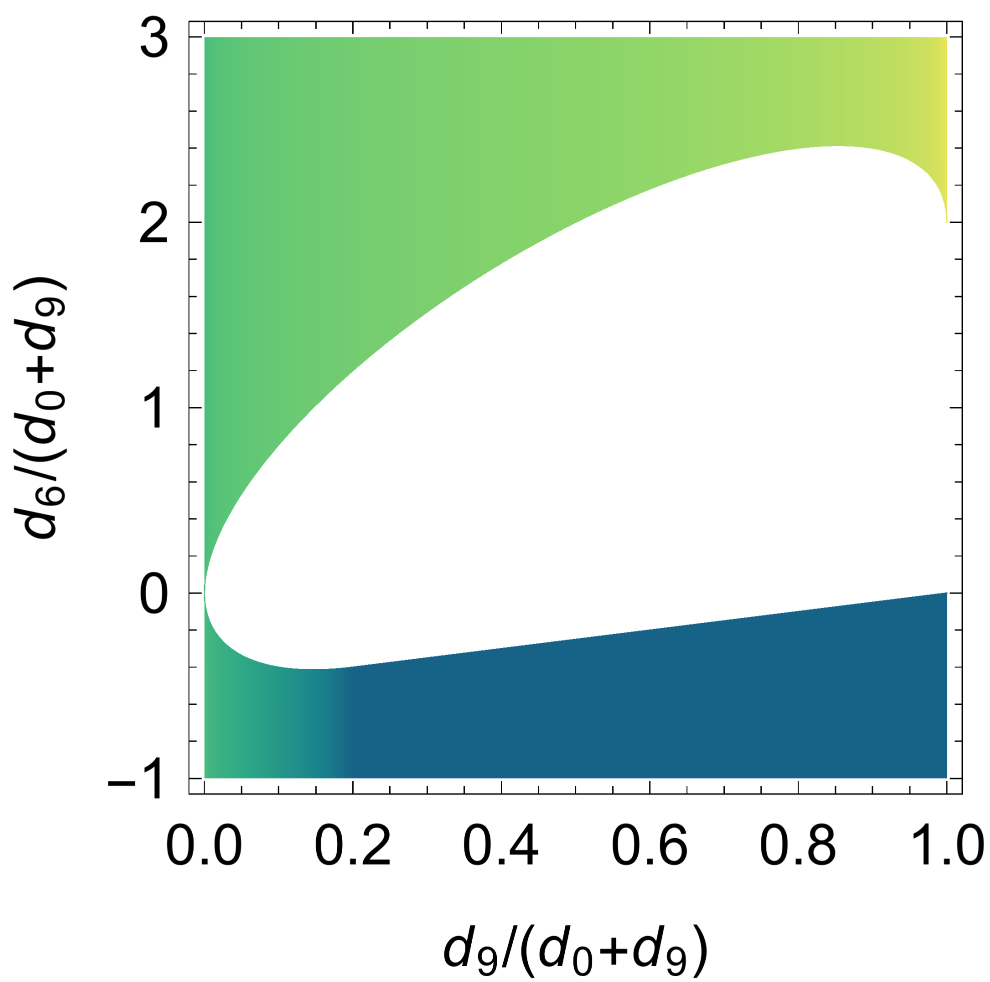
If $d_{0,9}$ are generated by some QFT states at scale m_ϕ, then $|d_6| \not\gtrsim 1/m_\phi^2$.
A similar conclusion follows from causality. Camanho et al. [1407.5597]

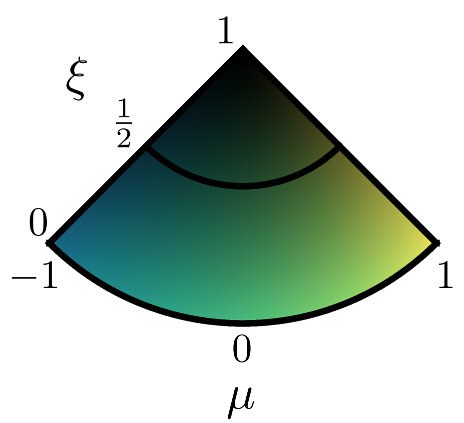
Consequences of EFT bounds

Let us explore consequences of our family of EFT bounds:

Bounding d_6:

Excluded regions:





UV examples

Cheung, Liu, GR [1903.09156]

Example completions

Scalar completion:

$$\mathcal{L} = \overline{\mathcal{L}} - \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - \frac{1}{2}m_\phi^2\phi^2 + \phi\left(a\kappa^{-1}R + \kappa b F_{\mu\nu}F^{\mu\nu}\right)$$

Wilson coefficients:

$$d_{1,\ldots,8} = \frac{1}{2m_\phi^2} \times (a^2, 0, 0, 2ab, 0, 0, b^2, 0)$$

$$(d_0, d_3, d_6, d_9) = \frac{2b^2}{m_\phi^2} \times (1, 0, 0, 0)$$

Entropy bound:

$$\Delta S(\xi, \mu, 0) = \frac{64\pi^2(1-\xi)^2 b^2 \mu^2}{5\kappa^2 m_\phi^2 \xi(1+\xi)} > 0$$

Example completions

Pseudoscalar completion:

$$\mathcal{L} = \overline{\mathcal{L}} - \frac{1}{2}\nabla_\mu \phi \nabla^\mu \phi - \frac{1}{2} m_\phi^2 \phi^2 + \kappa b \phi F_{\mu\nu} \widetilde{F}^{\mu\nu}$$

Wilson coefficients:

$$d_{1,\ldots,8} = \frac{b^2}{m_\phi^2} \times (0,0,0,0,0,0,-1,2)$$

$$(d_0, d_3, d_6, d_9) = \frac{2b^2}{m_\phi^2} \times (0,0,0,1)$$

Entropy bound:

$$\Delta S(\xi,\mu,0) = \frac{64\pi^2(1-\xi)^2 b^2 (1-\mu^2)}{5\kappa^2 m_\phi^2 \xi (1+\xi)} > 0$$

Example completions

Tensor completion:

$$\mathcal{L} = \overline{\mathcal{L}} + \mathcal{L}_{\text{FP}} + \kappa b \phi^{\mu\nu} \overline{T}_{\mu\nu} \quad \text{where} \quad \overline{T}_{\mu\nu} = F_{\mu\rho} F_{\nu}{}^{\rho} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}$$

where the Fierz-Pauli action is

$$\begin{aligned}
\mathcal{L}_{\text{FP}} = &- \frac{1}{2} \nabla_{\rho} \phi_{\mu\nu} \nabla^{\rho} \phi^{\mu\nu} + \nabla_{\mu} \phi_{\nu\rho} \nabla^{\nu} \phi^{\mu\rho} \\
&- \nabla_{\mu} \phi^{\mu\nu} \nabla_{\nu} \phi^{\rho}{}_{\rho} + \frac{1}{2} \nabla_{\rho} \phi^{\mu}{}_{\mu} \nabla^{\rho} \phi^{\nu}{}_{\nu} \\
&- \frac{1}{2} m_{\phi}^{2} (\phi_{\mu\nu} \phi^{\mu\nu} - \phi^{\mu}{}_{\mu} \phi^{\nu}{}_{\nu})
\end{aligned}$$

Wilson coefficients:

$$d_{1,\ldots,8} = \frac{b^2}{8 m_{\phi}^2} \times (0,0,0,0,0,0,-1,4)$$

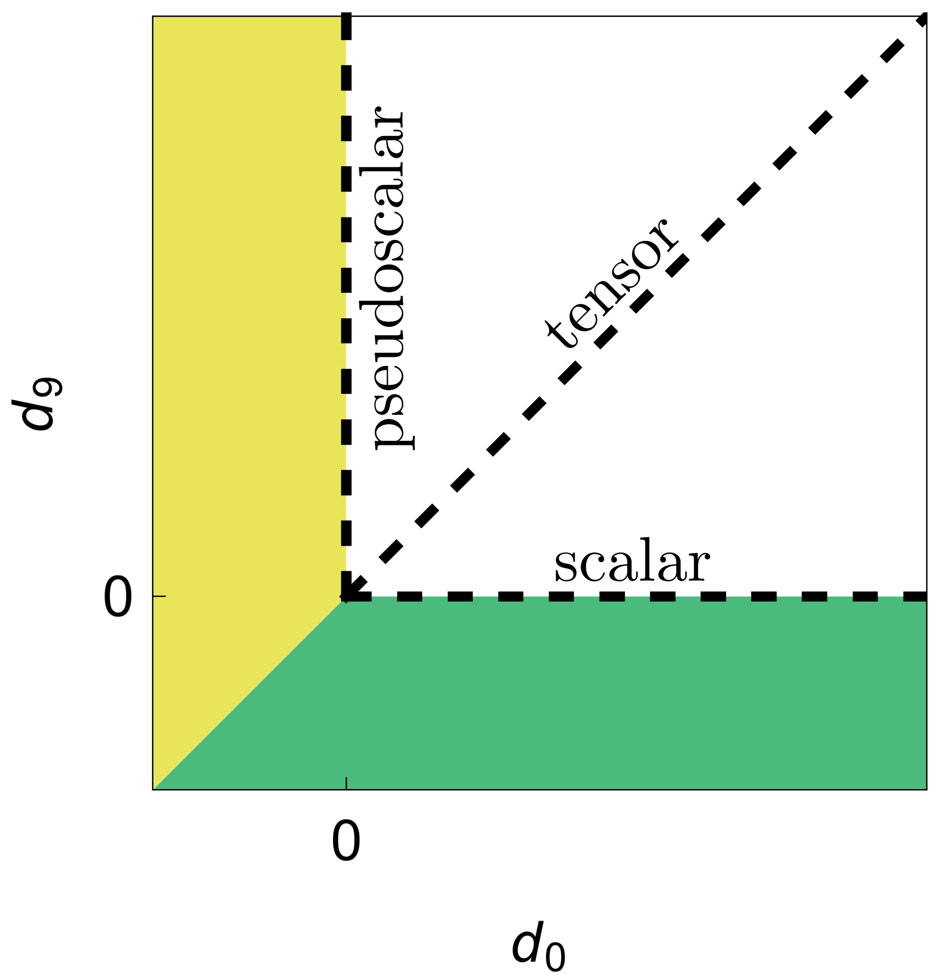
$$(d_0, d_3, d_6, d_9) = \frac{b^2}{2 m_{\phi}^2} \times (1,0,0,1)$$

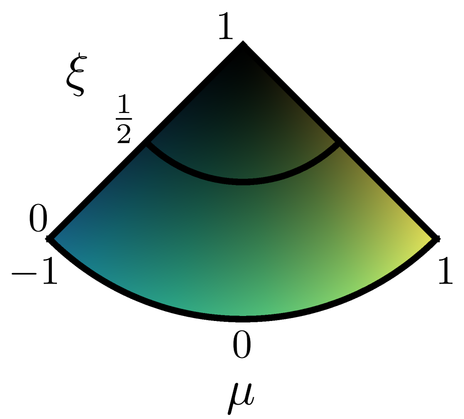
Entropy bound:

$$\Delta S(\xi, \mu, 0) = \frac{16 \pi^2 (1-\xi)^2 b^2}{5 \kappa^2 m_{\phi}^2 \xi (1+\xi)} > 0$$

Example completions

Summary of example completions:





Heterotic string

The low-energy action of the heterotic string is: Gross, Sloan (1987)

$$\mathcal{L} = \overline{\mathcal{L}} + \frac{\alpha' e^{-\kappa\phi/\sqrt{2}}}{16\kappa^2}\left(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right)$$
$$- \frac{3\alpha'\kappa^2 e^{-\kappa\phi/\sqrt{2}}}{64}\left(F_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} - 4F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu}\right)$$

If we somehow pin the dilaton to zero and then compactify down to $D = 4$, we have Wilson coefficients:

$$d_{1,\ldots,8} = \frac{\alpha'}{64} \times (4, -16, 4, 0, 0, 0, -3, 12)$$
$$(d_0, d_3, d_6, d_9) = \frac{\alpha'}{16} \times (3, 1, 0, 3)$$

Entropy bound:

$$\Delta S(\xi, \mu, 0) = \frac{2\pi^2\alpha'}{5\kappa^2\xi(1+\xi)}(3 + 4\xi + 13\xi^2) > 0$$

Weak Gravity Conjecture

Cheung, Liu, GR [1903.09156, 1801.08546]

Extremality

Again define the extremality parameter:

$$\zeta = \frac{\sqrt{a^2 + q^2 + \widetilde{q}^2}}{m} = \sqrt{1 - \xi^2}$$

- New allowed range is $\zeta \in [0, 1 + \Delta\zeta]$, where $\Delta\zeta = -\lim_{\zeta \to 1} \left. \frac{\Delta g^{rr}}{\partial_\zeta \overline{g}^{rr}} \right|_{r_\mathrm{H}}$
- At fixed $(m, a, q, \widetilde{q})$, the Boyer-Lindquist coordinate of the horizon shifts by:

$$\Delta r_\mathrm{H} = - \left. \frac{\Delta g^{rr}}{\partial_r \overline{g}^{rr}} \right|_{r_\mathrm{H}}$$

- In the extremal limit, $\partial_r \overline{g}^{rr} \to 0$, so ΔS is dominated by the shift in area:

$$\Delta S \to \frac{16\pi^2}{\kappa^2} r_\mathrm{H} \Delta r_\mathrm{H}$$

- We can relate Δr_H and $\Delta\zeta$ by computing $\partial_\zeta \overline{g}^{rr} / \partial_r \overline{g}^{rr} |_{\mathrm{KN}, r = r_\mathrm{H}} = m\zeta/\xi$ so we obtain:

$$\boxed{\Delta\zeta(\mu, \nu) = +\frac{\kappa^2}{16\pi^2 m^2} \lim_{\xi \to 0} [\xi \Delta S(\xi, \mu, \nu)]}$$

Self-sufficient black hole decay

Our extremality shift relation

$$\Delta\zeta(\mu,\nu) = +\frac{\kappa^2}{16\pi^2 m^2}\lim_{\xi\to 0}[\xi\Delta S(\xi,\mu,\nu)]$$

plus our entropy bound

$$\Delta S(\xi,\mu,\nu) > 0$$

for all $-1 \leq \mu \leq 1$ and $0 < \xi < \dfrac{1-3\nu^2}{2(1+\nu^2)}$

together imply that

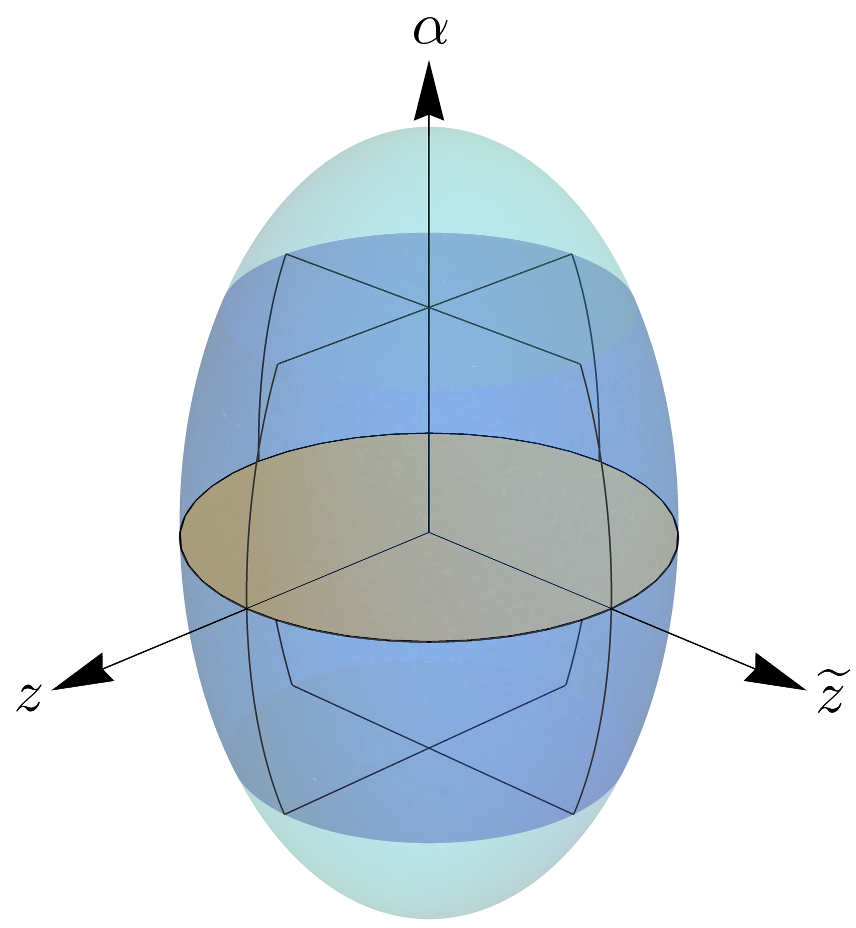
$$\Delta\zeta(\mu,\nu) > 0 \text{ for all } \mu \in [-1,1],\ \nu \in [0, 1/\sqrt{3})$$

In particular, $\Delta z > 0$ so:

Consistency of black hole entropy proves the Weak Gravity Conjecture.

Self-sufficient black hole decay

- Extremal black holes in barrel-shaped region with $|a|/m < 1/\sqrt{3}$ (required for thermodynamic stability) can decay to **other black holes nearly at rest**.
- Black holes with spin are kinematically allowed to decay to nonrotating dyonic ones via emission of gravitons.
- However, the fact that black holes can decay to other black holes at rest points to a principle of black hole self-sufficiency.



Discussion

- In this talk, we went beyond the leading-order modifications to the electric black hole, to see how the WGC and positivity bounds from analyticity/causality are connected in greater generality.
- We found that the quartic Riemann operators, which are positive by analyticity, give positive corrections to the extremality condition for all charged, spinning black holes.
- Investigated competing effects of R^4, F^2R^2, F^4 operators
- Analyticity of multi-particle scattering for P(X) theory gives a version of the WGC for BTZ black holes in 2+1 dimensions.
- Applied the $\Delta S > 0$ condition to bound the four-derivative (R^2, RF^2, F^4) corrections to dyonic, spinning black holes. Proved and generalized the WGC for tree-level completions of four-derivative operators.