School Admission When Students Have Diversity Preferences [1]

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Abstract

Schools and students care about the diversity of the school's student population. This paper diverges from the current literature and incorporates students' preferences for diversity into the School Admission model. The paper focuses on investigating the existence of stable matching in such a School Admission model. Specifically, I illustrate a school admission model in which students have preferences not only over schools but also over schools' student populations. This model extends the classical hedonic game framework by incorporating schools and capacity constraints into the coalition structures. The concepts of individual and group stability are redefined. In this case, the conditions of additive separability and symmetry, sufficient for ensuring the existence of individually stable matching in classical hedonic games, are no longer sufficient in this context. Additionally, I introduce the non-rival property, which, in conjunction with the top coalition property, ensures the existence of a unique group stable matching under strict preferences. Furthermore, A strategyproof algorithm that yields the unique group stable matching is proposed.

[1] This paper is advised by Professor Tilman Börgers

1 Introduction

This paper investigates the existence of stable matching under a School Admission model where students care about diversity at schools. As our society becomes more diverse and inclusive, schools have also shown a growing interest in fostering diversity. In the field of economic research, there have also been attempts to model such interest of schools into the school admission model. For example, Abdulkadiroglu et al. (2002) introduced a school choice model in which schools have choice functions that take both the diversity and ranking of students into account. However, there haven't been attempts to incorporate students' preferences over the diversity of schools' student populations into the school admission model. In reality, schools care about building a more diverse educational environment because they know that students benefit from such learning environments. Therefore, it's unreasonable for schools to be the only agents caring about diversity. This paper aims to construct a school admission model where students care about the diversity of schools.

There are various ways in which I can account for students' preferences for diversity. For example, it's possible to calculate a "diversity index" for schools and incorporate that into students' preferences. Taking such an approach might mean that I have to make students' preferences lexicographic. In this paper, I decided to take a more general approach. I simply consider the case where students have preferences over schools and subsets of students in that school. This approach makes fewer assumptions about what kind of "diversity" does students care about (Race, ethnicity, or gender, etc.) because it allows for preference over any characteristic of other students.

This paper begins by introducing the framework of the school admission model. In classical hedonic games, coalitions are formed between players who only care about the members of their coalition. In section 2, I extend the classical hedonic game model to the school admission context by incorporating schools and capacity constraints into the coalitions. Specifically, there has to be a school in each coalition of students, and the size of the coalition is constrained by the capacity of the school. I adopt the two notions of stability for classical hedonic games, individual stability and core stability, to the case of the school admission model. In section 3, I redefine individual stability to account for the size constraint of coalitions. I show that the two sufficient conditions for individual stability in classical hedonic games, additive separability and symmetry, are no longer sufficient. In section 4, I replace core stability with group stability, which is also about whether a group of agents can block. I show that the top coalition property, which guarantees the existence of core stable matching in hedonic games, no longer guarantees the existence of group stable matchings. I introduce a non-rival property, which, together with the top coalition property, ensures the existence of a unique group stable matching when preferences are strict. In section 5, a strategyproof algorithm that produces the unique group stable matching is presented.

2 Hedonic model where only students have preference

A hedonic game is a class of games where the players are matched into disjoint groups within themselves, and the preferences of the players only depend on the other members of their group. Our model is an extension of the classical hedonic game, and I adopted some frameworks of classical hedonic games from Bogomolnaia et al.(2002) paper: "The Stability of Hedonic Coalition Structures." Hedonic game models have a wide range of applications in economics, computer science, and other fields. Perhaps the most representative application is in the formation of societies or political groups. The players want to be matched into a coalition where members have similar political beliefs compared to themselves because the joint opinions of members of the party determine the political view of the party. Consequently, players only care about people who are in the same group as them (Bogomolnaia 2002). For the situation where students care about the diversity of schools in a school admission model, I want to consider the school admission problem under the framework of hedonic games. I'm interested in a hedonic game model where the students have preferences over both the schools and their peers at that school in order to address students' concerns about diversity.

The model of such a hedonic game contains a set of students $N = \{1, 2, 3..., n\}$ where $n \in \mathbb{N}$. I denote the set of schools as the set $S = \{s_1, s_2, ...s_k\}$ where $k \in \mathbb{N}$ .I divide the set $N \cup S$ into $k+1$ mutually disjoint groups . k groups have to contain exactly one school. These are the groups of students who are matched to schools. The capacity of school s_i is denoted by $c_i, c_i \in \mathbb{N}$, and the capacity of the school has to be weakly greater than the number of students in that school. There is exactly one group that contains no school. This group contain students who are not assigned to any school in the matching. Without loss of generality, I assume that students who are in this group don't care about other members in the group because they are not even in school. Therefore, for every agent $n \in N$, being in the group $\{N_{k+1}\}$ is considered as $\{n\}$ in their preferences.Note that this group can be an empty set when everyone is matched to a school. Denote π as a coalition structure, π is a set that contains $\{\{s_1, N_1\}, \{s_2, N_2\}, ...\{s_k, N_k\}, \{N_{k+1}\}\}$.I have $\bigcup_{n=1}^{l} N_n = N$ and $\forall n, m \leq l, s.t\ n \neq m, N_n \cap N_m = \emptyset$. Define $\pi(n)$ where $n \in N$ as all elements of the coalition that n is in, including himself. Each agent $n \in N$ has a preference $\succsim_n$ over $\{n\}$ and all possible sets $\{s_i, V\}$ $\forall i \in \{1, 2...k\}, V \subseteq N, s.t\ n \in V$.I denote the set of all possible alternatives for agent n as $S_n = \{\{s, V\} : s \in S, V \subseteq N, s.t\ n \in V.\}$.

Two major differences exist between the school admission model and a classical hedonic game model. It ultimately comes from the addition of schools. Firstly, in classical hedonic games, agents only have preference over groups that are formed within themselves. However, in the school admission model, the students not only have preferences over the group of students they are matched with but also care about what schools their coalition is in. Moreover, in classical hedonic games, there are no constraints to the sizes of the coalitions. However, in the school admission model, the size of coalitions is constrained by the schools they are matched to. For example, if a school has a capacity of three students, then the coalition formed in that school can contain at most three students. These differences in the model framework necessitate modifications to the classical notions of stability for their application in the school admission model.

3 Individual Stability

Given that our model also falls within the framework of hedonic games, it's desirable that I extend the existing notions of stabilities to the context of our model. For convenience, I want to define the set of open coalitions.

Definition 1. *For any coalition structure π, define the set of open coalitions $O_\pi \subseteq \pi$, where $O_\pi = \{x \in \pi : \exists s_i \in x, where\ s_i \in S, and\ |x \backslash \{s_i\}| < c_i\} \cup \{y \in \pi : \nexists s_i \in x, where\ s_i \in S\}$. This set $O_\pi$is basically all schools that are not at full capacity and the set of students who are not matched to any schools. O_π^c denotes the complement of O_π.*

Firstly, I consider the cases where only one agent is blocking. In classical hedonic games, a matching is individually stable as long as it doesn't have any agent i, who wants to deviate from his current coalition to a coalition where he is welcomed. This means that everyone in the matching that he deviates to must be weakly better off. If I follow that exact framework, a student i could block to deviate to another coalition if every student is weakly better off when he joins. If the coalition is open, the situation is trivial and identical to that of a classical hedonic game. However, if the coalition is not open, following the same definition means that a student could only block to deviate to a non-open coalition if there is some student j in that coalition who he could replace so that both of them are better off. The complication comes from caring about the student who is expelled from the coalition. If I want to know whether he is better off, I need to know his new matching after he is expelled. However, it's unclear what will happen to him because I don't have any specifications for how student j should deviate, and even if I allow him to self-select into the coalition that he most prefers, I need to consider whether I care about the preferences of people in that coalition. If I don't consider the preferences of people in that coalition, it seems like I'm contradicting my definition of blocking groups. Initially, I allow student i to block only if everyone in the new coalition is weakly better off, but this requirement does not apply to agent j. If I do require that student j moves to a coalition in which all students in that coalition are weakly better off, the definition of blocking group coincides with the definition of pareto improvement as every agent is better off through i's deviation.

Therefore, to avoid such complications, I could either just not allow a student to deviate to a coalition that is non-open, or I could not care about the student j who is expelled. If I only allow a student to deviate to an open coalition, the definition of stability becomes rather useless for coalition structures in which all schools are at full capacity. Consider any coalition structure where all schools are at full capacity. In this scenario, the coalitions will be stable as long as everyone prefers their current coalition to disenrolling from school. Any arbitrary matching that matches students randomly to schools so that every school is at full capacity will be stable. Consequently, I have arrived at the plausible definition of blocking groups for individual stability where I don't care about the student who is expelled.

Definition 2. *A matching π is individually stable if and only if*

1) there doesn't exist any agent n s.t $n \in N$, and a coalition $s \in O_\pi$ s.t $n \notin s$, $s \cup \{n\} \succ_n \pi(n)$, and $\forall m \in s \cap N, s \cup \{n\} \succsim_m s$.

and 2)there doesn't exist any agent n s.t $n \in N$, and a coalition $s \in O_\pi^c$ s.t $n \notin s$ and $\exists p \in s \cap N$, $(s \backslash \{p\}) \cup \{n\} \succ_n \pi(n)$, and $\forall m \in (s \backslash \{p\}) \cap N, (s \backslash \{p\}) \cup \{n\} \succsim_m s$.

A blocking agent n under individual stability is a student who either wants to go to another school or disenroll from school because he will be strictly better off. If he is going to a school that is not at full capacity, it must be that other students in his new school are not worse off. If the school he wants to go to is at full capacity, there has to be a student p in the school who can be kicked out, and agent i can take his seat so that everyone else in that school is weakly better off.

For classical hedonic games, Bogomolnaia and Jackson(2002, Proposition 2) proved that symmetry and additive separability of the preferences ensure the existence of individually stable matchings. However, their proof doesn't apply to our school admission model. To prove this, I want to first define the restrictions that they imposed.

Definition 3. *A player i's preference $\succsim_i$ is additively separable if there exists a utility function for all students $i \in N$ such that $u_i : N \cup S \to \mathbb{R}. \forall j \in N, u_i(j)$ represents the utility that agent i gain from being in the same coalition as $j. \forall k \in S, u_i(k)$ represents the utility that agent i gain from being in school k. $(s,t) \in \succsim_i$,* *iff* $\sum_{j \in s} u_i(j) \geq \sum_{j \in t} u_i(j)$. *Without loss of generality,* $u_i(i) = 0$.

In the context of our model, this means that a student assigns a specific utility to being in the same matching with each of the other agents. He also gains a specific utility from being in each of the schools. An implicit assumption is that these utilities are also independent of each other because the utility of i for being in the same matching with any agent j is only a function of j. The preference of any student can be represented as a sum of the utility that he gains from all elements of his coalition which could contain students and a school.

Definition 4. *A set of additively separable preferences* $\succsim = \{\succsim_i : i \in N\}$ *satisfy symmetry iff* $\forall i, j \in N, u_i(j) = u_j(i)$.

Symmetry requires that for any pair of students i, j, the utility that i gains from having j in his matching is equal to the utility that j gains from having i in his matching.

Lemma 1. *For the school admission model, additive separability and symmetry do not guarantee the existence of an individually stable matching.*

Proof. To see that additive separability and symmetry don't ensure the existence of individually stable matchings, consider an example where there are two schools $\{A, B\}$ and three students $\{1, 2, 3\}$. The capacity of school A is 2 and the capacity of school B is 1. Their preference profiles are described as below:

$\succsim_1$: $\{1, 2, A\} \sim \{1, 3, A\} \succ \{1, A\} \succ \{1, B\} \sim \{1\}$

$\succsim_2$: $\{1, 2, A\} \succ \{2, A\} \sim \{2, 3, A\} \succ \{2, B\} \sim \{2\}$

$\succsim_3$: $\{1, 3, A\} \succ \{3, A\} \sim \{2, 3, A\} \succ \{3, B\} \sim \{3\}$

This preference is additively separable and symmetric. It can be represented by utility indexes u_{ij} $s.t$ $u_{ij} = u_{ji}$. For example, I can have $u_{12} = u_{13} = 1$,$u_{23} = 0$,$u_{1A} = u_{2A} = u_{3A} = 1$,$u_{1B} = u_{2B} = u_{3B} = 0$, and $u_{11} = u_{22} = u_{33} = 0$. There is no individually stable matching. If 1 is in school A, he could either be matched with 2 or 3 or himself, and in any case, either 2 or 3 will form a blocking pair with 1. If 1 is not in school A, he could always form a blocking pair with any agent in A. Therefore, an individually stable matching doesn't exist.

□

Evidently, it can also be shown that Bogomolnaia and Jackson's proof (2002) doesn't work. Firstly, note that since the preferences are symmetric, $\forall i, j \in N, u_i(j) = u_j(i)$. For simplicity, I denote $u_i(j)$ as u_{ij}. For a school $s \in S$, I denote the students' utility from being in schools s as u_{is}. According to Bogomolnaia et.al's proof , a coalition structure π that maximzes $\sum_{kj:\exists s\in\pi,k,j\in s\cap N} u_{kj} + \sum_{k\in N} u_{k(\pi(k)\cap S)}$ will be stable. It relies on the fact that the sum will strictly increase if I allow a blocking agent to deviate to his desired matching. However, this is no longer true for the school admission model. For any agent $i \in N$ and any two coalitions $\pi(i)$ and any $m \in O_\pi^c$that is at full capacity (ie: $m \notin O_\pi$), agent $i \in N$ would be blocking and want to deviate to m if there exists an agent $p \in N$ such that $\pi(p) = m$ *and* $(m\backslash\{p\}) \cup \{i\} \succ_i \pi(i),$ *and* $\forall j \in (m\backslash\{p\}) \cap N, (m\backslash\{p\}) \cup \{i\} \succsim_j m$. This is equivalent to saying$\sum_{j\in m\backslash\{p\}} u_{ij} > \sum_{j\in\pi(i)} u_{ij}$ and $\sum_{j\in m\backslash\{p\}} u_{ij} \geq \sum_{j\in m\backslash\{p\}} u_{pj}$. However, I no longer have $\sum_{kj:\exists s\in\pi,k,j\in s\cap N} u_{kj} + \sum_{k\in N} u_{k(\pi(k)\cap S)}$ will strictly increase if the agent i could profitably deviate. Suppose i profitably deviates to a non-open coalition from π to μ. $(\sum_{k,j:\exists s\in\mu,k,j\in s\cap N} u_{kj} + \sum_{k\in N} u_{k(\mu(k)\cap S)}) - (\sum_{k,j:\exists s\in\pi,k,j\in s\cap N} u_{kj} + \sum_{k\in N} u_{k(\pi(k)\cap S)})$ is the change in sum of utility of the all student pairs k, j that are within the same coalition from matching π to μ. I know that this term will equal to $\sum_{j\in m\backslash\{p\}} u_{ij} - \sum_{j\in\pi(i)} u_{ij} - \sum_{j\in m\backslash\{p\}} u_{pj} + \sum_{j\in\mu(p)} u_{pj}$. Firstly, I can not know the exact sign of this term without knowing where p goes. In fact, I don't know where p goes because I don't care about what happens to p in the definition of individually unstable blocking groups. Only if p is better off in μ

do I know that the term is positive because I have $\sum_{j \in m \setminus \{p\}} u_{ij} > \sum_{j \in \pi(i)} u_{ij}$ and $\sum_{j \in m \setminus \{p\}} u_{pj} < \sum_{j \in \mu(p)} u_{pj}$. However, that is not always guaranteed. Therefore, I do not know the exact change in utility when one blocking group is resolved, so I can't prove the existence of individually stable matching by maximizing the sum of utilities.

As I illustrated, the main reason the Bogomolnaia and Jackson proof (2002) does not work is due to the capacity constraint for coalitions imposed by the schools, which is also one of the major differences between the school admission model and a classical hedonic game. Consequently, it is possible to extend the proof of Bogomolnaia and Jackson's proof (2002) if I consider the case where schools have infinite capacities. Since there are no capacity constraints, every coalition would be open.

Proposition 1. *If the preferences are additively separable and symmetric, and there are no capacity constraint, an individually stable matching exists.*

Proof. It is sufficient to show that under symmetry and additive separability, there exists a matching where no agent can profitably deviate. For any agent $i \in N$ and any two coalitions $\pi(i)$ and any $m \in O_\pi$, if the coalition m is not at full capacity, then i is willing to deviate from $\pi(i)$ to m if and only if $m \cup \{i\} \succ_i \pi(i) \iff \sum_{j \in m} u_{ij} > \sum_{j \in \pi(i)} u_{ij}$. The two sums include the utility of all elements in i's coalition, and j could be either a school or a student. Consider the sum $\sum_{k,j:\exists s \in \pi, k,j \in s \cap N} u_{kj} + \sum_{k \in N} u_{k(\pi(k) \cap S)}$, which sums the utility between all student pairs k, j that are within the same coalition and the utility of all agents for being at the school he is assigned to. No agent could profitably deviate to another coalition if the sum is maximized. For any coalition structures π, if agent i can profitably deviate to another coalition, there must exist a coalition structure μ, and an agent i could profitably deviate if $\sum_{j \in \mu(i)} u_{ij} > \sum_{j \in \pi(i)} u_{ij}$. This also means that$\sum_{k,j:\exists s \in \mu, k,j \in s \cap N} u_{kj} + \sum_{k \in N} u_{k(\mu(k) \cap S)} >$

$\sum_{k,j:\exists s\in\pi,k,j\in s\cap N} u_{kj} + \sum_{k\in N} u_{k(\pi(k)\cap S)}$ because for all terms u_{kj} such that $k,j\neq i$ the utility level doesn't change from μ to π due to additive separability. For π ,the terms u_{kj}such that $k\ or\ j = i$ are exactly the terms $\sum_{j\in\pi(i)} u_{ij}$, and the same holds for μ. Therefore, If an agent can profitably deviate to another coalition from coalition structure π, there must exist some coalition structure μ where the sum $\sum_{k,j:\exists s\in\mu,k,j\in s\cap N} u_{kj} + \sum_{k\in N} u_{k(\mu(k)\cap S)}$ is bigger than the sum for π. If the sum $\sum_{kj:\exists s\in\pi,k,j\in s\cap N} u_{kj} + \sum_{k\in N} u_{k(\pi(k)\cap S)}$ is maximized, there exists a matching π where no agent can profitably deviate to another coalition. There are finitely many possible coalition structures, the set of possible values of $\sum_{kj:\exists s\in\pi,k,j\in s\cap N} u_{kj} + \sum_{k\in N} u_{k(\pi(k)\cap S)}$ is also finite, so $max \sum_{kj:\exists s\in\pi,k,j\in s\cap N} u_{kj} + \sum_{k\in N} u_{k(\pi(k)\cap S)}$ exists. Therefore, there exists a coalition structure π where no agents will profitably deviate. This is sufficient to ensure the existence of an individually stable matching when there are no capacity constraints in these games.

$\square$

4 Group Stability

Definition 5. *A matching π is group stable if and only if there doesn't exist any blocking group B s.t $B\subseteq N, \exists s_i\in S\cup\{\emptyset\}, c_i\geq |B|, s.t\ \forall n\in B, \{s_i,B\}\succsim_n \pi(n), \forall n\in B\backslash\{B\cap s_i\}, \{s_i,B\}\succ_n \pi(n)$.*

A blocking group in terms of group stability is a group of agents that weakly prefer forming another coalition within themselves to their current coalition. For the students in this coalition who want to move to a different school, they need to strictly prefer the coalition formed by the blocking group to their current coalition. Notice that this blocking group must include all agents in the coalition that the blocking agents prefer to their current coalition. Note that group stability can be viewed as a direct extension of individual stability by allowing more than one student to block. In fact, group stability implies individual stability.

Lemma 2. *Group stable matchings are always individually stable.*

Proof. If a coalition structure μ is not individually stable, then there exists a blocking group such that one student strictly prefers being matched with everyone else in the blocking group and their school to his current school and classmates. Therefore, μ is not group stable.

$\square$

In a classical hedonic game where agents have preferences only over their group members and are divided into groups, Banerjee. et al(2001) proved that the top coalition property is sufficient to guarantee the existence of core stable matchings in Theorem 2. Core stability requires that there doesn't exist a blocking group such that every agent preferres the blocking group to their current matching. However, for a blocking group in group stability, I encorporated schools into the blocking group. Moreover, I only require students that move to a different school to be strictly better off.

In this paper, I will also use this property to ensure the existence of group stable matching, and it's necessary to extend its definition to the context of our model.

Definition 6. *A hedonic game satisfies the top coalition property iff* $\forall a \subseteq N \cup S, a \neq \emptyset$*,*$\exists z \subseteq a, z \neq \emptyset,$ $|z \cap S| \leq 1, s.t\ if\ \exists s_i \in z, s.t\ s_i \in S,\ then\ c_i \geq |z|$*,and* $\forall n \in z \cap N, z \succsim_n x\ \forall x \subseteq a\ s.t\ n \in x.$

(note that the set $\{s_i, z\}$ *is referred to as the top coalition of subset* a*)*

This property requires that for all nonempty subsets a of all players $N \cup S$, there must be a non-empty subset z of a ,called the top coalition. Every student n in this top coalition must weakly prefer this coalition over any other possible coalition that they can form with agents in a. This top coalition can include at most one school. If this top coalition includes a school, the capacity of the school s_i should be weakly smaller than the size of the coalition. Note that there could be more than one top coalition in a, but I only require that there is at least one top coalition in every a. In a classical hedonic game, it can be shown that a game that satisfies this assumption and has strict preferences will always have core stable matchings. I have also adopted this assumption as one of the preconditions to ensure the existence of a group stable matching.

Definition 7. *A hedonic game satisfies the non-rival property iff for any top coalitions a, b from any subsets of N ,$a \cap b = \emptyset$.*

Non-rival property basically ensures that there are not two groups of students who want to go to the same school with their coalition. If this happens, a group stable matching could never exist because either group will always block. Below is an example of a preference that satisfy these two conditions where there are two schools $\{A, B\}$ and two students $\{1, 2\}$. The capacity of school A is 2 and the capacity of school B is 1. Their preference profiles are described as below:

$\succsim_1$: $\{1, 2, A\} \succ \{1, A\} \succ \{1\} \succ \{1, B\}$

$\succsim_2$: $\{1, 2, A\} \succ \{2, B\} \succ \{2, A\} \succ \{2\}$

This set of preferences satisfy both the non-rival property and the top coalition property. The unique group stable coalition is {(1,2,A),(B)}

Notice that if the set of preferences doesn't satisfy the non rival property, a group stable matching will not exist:

$\succsim_1$: $\{1, A\} \succ \{1, 2, A\} \succ \{1\} \succ \{1, B\}$

$\succsim_2$: $\{2, A\} \succ \{1, 2, A\} \succ \{2, B\} \succ \{2\}$

I can trivially show that these preferences satisfy the top coalition property. However, for the set {1,2,A,B}, there are two top coalitions:$\{1, A\}, \{2, A\}$. Obviously, their intersection is not empty. Therefore, either of the top coalition will always block for any arbitrary coalition structure.

Theorem 1. *A unique group stable matching exists if preferences are strict, and the top coalition and non-rival properties are satisfied.*

Proof. The statement will be proved in a similar manner to Banerjee et. al(2001)'s proof. The main differences come from the addition of schools. Agents not only have preferences over possible subsets of students, they also have preferences over schools. Since coalitions are viewed as groups of students at a school, the same group of students that are assigned to different schools will be considered as different coalitions. Schools also have capacity constraints, so that some coalitions may not exceed certain sizes. While only the top coalition property is sufficient for the existence of core stable matchings in classical hedonic games, it is no longer sufficient in this context unless the non-rival property is also satisfied.

Deonte the set of all top coalitions in f as $T_f = \{x_{f_1}, x_{f_2} ..., x_{f_k}\}, f \subseteq N$. For this matching algorithm, I start with the entire set of students and schools $N_1 = N \cup S$ and match all the top coalitions in N_1, and I can do this because the intersection between any of these coalitions is empty. Then I consider the set $N_2 = N_1 \backslash \bigcup_{x \in T_N} x$, and match all top coalitions in N_2. For step m, I match all top coalitions in the set $N_m = N_{m-1} \backslash \bigcup_{x \in T_{N_{m-1}}} x$. Continue inductively until I have $N_n = \emptyset$ for some $n \in \mathbb{N}$. I know that such an algorithm terminates in finite steps because N is a finite set, and at least one top coalition exists in every step. The produced matching will be stable because all students in top coalitions of N_1 are in their most preferred matchings, so they will not form a blocking group. There are only two cases where students in top coalitions of N_2 could be better off. Firstly, they could be better off forming coalitions with agents in T_{N_1} , but these students are in their most preferred matchings. Otherwise, they might be better off if they form a coalition with agents in $N_2$to go to schools that are occupied by previous top coalitions; however, notice that this would violate the non-rival property. Subsequently, agents in top coalitions of N_i, will only be better off forming a coalition with students in top coalitions of N_j $s.t$ $j < i$. Therefore, this matching is stable.

This matching is also unique. I can prove this by contradiction. Suppose another matching μ is also stable, it must contain all coalitions in T_{N_1}because any coalition in T_{N_1}that is not in μ would form a blocking group. By the top coalition property, all agents in coalitions in T_{N_1} strictly prefer their coalition in T_{N_1} than any other coalitions that they could form with students in N and schools in S. If μ has all coalition in T_{N_1}, then all agents in T_{N_1}will never block. It must also contain all coalitions in T_{N_2}, otherwise coalitions in T_{N_2} will block. Continue inductively, I have that matching μ must be the matching that I construct above.

□

5 Strategyproofness of the top coalition algorithm

Now let's consider an algorithm that is based on the top coalition property. Suppose people have strict preferences, and their preferences satisfy the top coalition and non-rival property, then this algorithm produces a unique group stable matching. The first stage of the algorithm begins by allowing every student to submit a proposal to their most preferred coalition.This includes a school and a group of students.If there exists a proposed coalition for which all students in that coalition proposed that coalition, the proposed coalition is formed.The students who are not matched in a coalition will proceed to the next stage. For the second stage, I are left with a subset of all students N for whom the initial proposal is not accepted. I allow those students whose proposal in the first stage is not feasible anymore to submit a new proposal(ie.

some agents in their previous proposal are matched with others). The implicit rule is that agents whose proposals were not accepted in the previous round can not change their proposal unless their proposal becomes unfeasible. This rule should be satisfied by all rational agents. Suppose they choose coalition a as their most preferred coalition in the previous round, in the next round, coalition a should still be their most preferred coalition if it's available because their set of choices strictly decreases from the previous stage. All proposed coalitions that are proposed by all of its members are formed. Then I begin the third stage of proposals with all students whose proposals were not successful and continue as I did in the previous stages.

This algorithm will always end in finite steps because I have that a top coalition exists in any subset of N, so at least one proposed coalition will be formed in every stage. Therefore, the number of students that are left unmatched in each stage is strictly decreasing. Obviously, this algorithm produces a group stable matching because it matches the top coalitions in every stage. The intuition is exactly the same as the previous proof for top coalition property.All the students who were matched in the first stage are not willing to deviate because they are already in their most preferred matching. All the students who are matched in the second stage are matched with their most preferred coalition in the second stage, and they are only better off if they are matched with students who are matched in the first stage. For any stage $n > 1$, the students are only better off if they are matched with students who formed coalitions in the previous stages. Therefore, the matching is stable.It's desirable for us to asses whether this algorithm is strategy-proof. An algorithm is said to be strategyproof if it is always a weakly dominant strategy for any player to reveal their true preference regardless of what others do.

Theorem 2. *The top coalition algorithm is strategy-proof.*

Proof. I want to show that it's a best response for any player $n \in N$ to reveal his true preference given any arbitrary strategy profile $\succsim_{-n}$ of all other players. This is equivalent to saying that it's a best response for any player $n \in N$ to propose his top coalition in any stage for any $\succsim_{-n}$. I denote the strategy where player n always proposes his top coalitions as $\succsim_{t(n)}$ and denote the coalition that player n will achieve from this strategy as t. By contradiction, suppose that there exists a player n and a strategy profile $\succsim_{-n}$ of all other players, for which reporting top coalition is not player n's best response. It must be that there exists a set of strategies $\succsim_{x(n)}$ of n such that n is matched to some coalition x and $x \succ_n t$.

Given the existence of strategies $\succsim_{x(n)}$ of n, if we fix $\succsim_{-n}$ for other players, when n plays $\succsim_{t(n)}$, we can show that n can choose to propose x in any period, and he will be matched. Note that this also means that all members of x will not be matched until n is matched because everyone in x except n proposed x, and they will not be able to change their proposal until n is matched. To show this, consider the game where n plays a strategy in set $\succsim_{x(n)}$ such that n is matched to x in the earliest stage, there must exist a stage o where n proposes x and is matched. o is the earliest stage for which everyone in x proposes x.

First, we want to show that everyone in x will stay unmatched until o if n plays $\succsim_{t(n)}$. Since only n changed strategy, all proposed coalitions without n before stage o for the game where n plays $\succsim_{x(n)}$ will still form in the game where n plays $\succsim_{t(n)}$. Likewise, those that didn't form will still not form. n will also not form any coalition because, by definition , we know that n is proposing his top coalition in every stage. If he is matched, it means $t \succ_n x$ which contradicts our assumption. Therefore, all members of x will stay unmatched until stage o in the game where n plays $\succsim_{t(n)}$.

This is sufficient to show that when n is playing $\succsim_{t(n)}$, he can deviate to propose x in any period to be matched. Since all other members of x will stay unmatched until stage o and will propose x in stage o, n can propose x in any stage before o and he will be matched in stage o. He can also propose x in any stage after o and get matched in that stage because all members in x except n proposed x in stage o.

Suppose n is playing $\succsim_{t(n)}$, in any stage, if he proposes a top coalition c , it must be that $c \succ_n x$ because x is also feasible. He must not be matched to c because if he does, we have $c = t \succ_n x$. We have reached a contradiction because we get that n can't be matched in any stage while, by assumption, we have n is matched to some top coalition t. Therefore, such x doesn't exist, and $\succsim_{t(n)}$ is n's best response for any arbitrary strategy profile $\succsim_{-n}$ of all other players.

$\square$

6 Conclusion

To address students' preferences regarding diversity, this paper investigates the school admission model within the framework of a hedonic game. With slight modifications, I have defined the concepts of individual stability and group stability, analogous to individual stability and core stability in a classical hedonic game. Although additive separability and symmetry are sufficient to guarantee the existence of individually stable matchings in classical hedonic games, these conditions are no longer sufficient for the school admission model due to capacity constraints. However, Banerjee et al. (2001)'s proof for the existence of core stable matchings can be extended to ensure the existence of group stable matchings by imposing a new restriction: the non-rival property. An algorithm that is strategy-proof and produces such group stable matchings has also been shown to exist.

Further extension of the model is desirable, as the schools' preferences have not been incorporated. However, since the schools also have preferences over subsets of students, it's likely that the same results for the existence of stable matchings hold.

7 References

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