Name:________________________________

Exam Number:_______

University of Michigan Physics Department
Graduate Qualifying Examination

Part II: Modern Physics

Saturday 14 May 2016 9:30 am – 2:30 pm

This is a closed book exam, but a number of useful quantities and formulas are provided in the front of the exam. (**Note that this list is more extensive than in past years.**) If you need to make an assumption or estimate, indicate it clearly. Show your work in an organized manner to receive partial credit for it. Answer the questions directly in this exam booklet. If you need more space than there is under the problem, continue on the back of the page or on additional blank pages that the proctor will provide. <u>Please clearly indicate if you continue your answer on another page. Label additional blank pages with your exam number, found at the upper right of this page (but not with your name). Also clearly state the problem number and "page x of y" (if there is more than one additional page for a given question).</u>

You must answer the first 8 required questions and 2 of the 4 optional questions. Indicate which of the latter you wish us to grade (e.g. by circling the question number). We will only grade the indicated optional questions. Good luck!!

Some integrals and series expansions

$$\int_{-\infty}^{\infty} \exp(-\alpha x^2)\,\mathrm{d}x = \sqrt{\frac{\pi}{\alpha}}$$

$$\int_{-\infty}^{\infty} x^2 \exp(-\alpha x^2)\,\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\exp(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots$$

Some Fundamental Constants

speed of light $c = 2.998 \times 10^{8}$ m/s

proton charge $e = 1.602 \times 10^{-19}$ C

Planck's constant $\hbar = 6.626 \times 10^{-34}$ J·s $= 4.136 \times 10^{-15}$ eV·s

Rydberg constant $R_{\infty} = 1.097 \times 10^{7}\,\mathrm{m}^{-1}$

Coulomb constant $k = (4\pi\varepsilon_0)^{-1} = 8.988 \times 10^{9}$ N·m^2 / C^2

vacuum permeability $\mu_0 = 4\pi \times 10^{-7}$ T·m/A

universal gas constant $R = 8.3$ J / K·mol

Avogadro′s number $\mathrm{N_A} = 6.02 \times 10^{23}\,\mathrm{mol^{-1}}$

Boltzmann′s constant $\mathrm{k_B = R/N_A} = 1.38 \times 10^{-23}$ J/K$= 8.617 \times 10^{-5}$ eV/K

Stefan-Boltzmann constant $\sigma = 5.67 \times 10^{-8}\,\mathrm{W / m^2K^4}$

radius of the sun $R_{\mathrm{sun}} = 6.96 \times 10^{8}$ m

radius of the earth $R_{\mathrm{earth}} = 6.37 \times 10^{6}$ m

radius of the moon $R_{\mathrm{moon}} = 1.74 \times 10^{6}$ m

gravitational constant $G = 6.67 \times 10^{-11}\,\mathrm{m^3 / (kg{\cdot}s^2)}$

Required (do all of the first 8 problems)

1. (**Quantum Mechanics**) In one and two dimensions, any attractive potential, no matter how weak, will admit at least one bound state. However, the same is not true in three dimensions. Consider a particle of mass m in a finite spherical well in three dimensions with potential

$$V(r) = \begin{cases} -V_0 & r < a \\ 0 & r > a \end{cases}$$

Find the minimum value of V_0 for which a bound state can exist. (You can assume this will occur for angular momentum $l = 0$.)

2. (**Quantum Mechanics**) A forced harmonic oscillator is described by the Hamiltonian

$$H = (a^\dagger a + \tfrac{1}{2})\hbar\omega + f(t)a + f^*(t)a^\dagger$$

where a and $a^\dagger$ are the usual annihilation and creation operators satisfying $[a, a^\dagger] = 1$. Here $f(t)$ is a time-dependent forcing function and $f^*(t)$ is its complex conjugate. The harmonic oscillator is initially in its ground state, and then the forcing function is turned on at time t_0.

a) Find an expression for the transition probability $P_{n\leftarrow 0}$ from the ground state to the n-th excited state in first-order time-dependent perturbation theory.

b) What is the probability for the oscillator to be found in an excited state at time $t = \infty$ if the forcing function

$$f(t) = f_0 e^{-t^2/2\tau^2}$$

(where f_0 and τ are constants) is turned on at time $t_0 = -\infty$?

3. (**Quantum Mechanics**) Consider a localized electron (spin operator $\vec{S} = \frac{1}{2}\hbar\vec{\sigma}$, where $\vec{\sigma}$ are the Pauli matrices) with magnetic moment $\vec{\mu}$ and gyromagnetic ratio g, whose Hamiltonian in the presence of an external magnetic field $\vec{B}$ is:

$$H = -\vec{\mu} \cdot \vec{B} = \frac{eg\hbar}{4mc}\vec{\sigma} \cdot \vec{B}.$$

The direction of the magnetic field is taken to define the z axis. Suppose that at time $t = 0$ the spin is an eigenstate of S_x with eigenvalue $\hbar/2$. What is the expectation value of S_x and S_y at a later time t? What is the physical interpretation of your result?

4. (**Quantum Mechanics**) A particle of mass m at $t = 0$ is in the ground state of an infinite square well of width L. Suddenly the well expands with the right wall moving from $x = L$ to $x = 3L$, leaving the wavefunction momentarily undisturbed.
a) If the energy of the particle is is now measured, what is the most probable result and what is the probability of obtaining this result?
b) What does the wavefunction of the particle look like after a time t? Is it still an energy eigenstate?

The energy eigenvalues and eigenfunctions of a particle of mass m in an infinite square well of width a are given by

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}, \quad n = 1, 2, ...$$

$$\psi_n = \sqrt{\frac{2}{a}} \sin\frac{n\pi x}{a}, \quad 0 \leq x \leq a.$$

5. (**Statistical Mechanics**) Consider a 1D quantum harmonic oscillator with eigen-energies $E_n = (n + 1/2)\hbar\omega$, where $n = 0,\ 1,\ 2,\ \ldots$

a) At temperature T, what is the probability for the quantum oscillator to be at its ground state

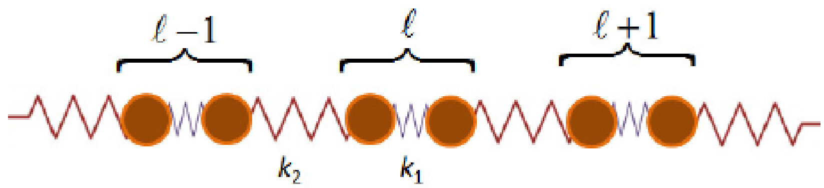
b) Prove that at high temperature, the probability is $P \approx \hbar\omega/(k_B T)$

c) Prove that at low temperature, the probability is $P \approx 1$

6. (**Statistical Mechanics**) A system consists of N weakly interacting particles at a temperature T. Each particle has a mass m and performs one dimensional oscillations about its equilibrium position. Assuming the validity of classical statistical mechanics calculate the heat capacity of this system for each of the following cases:
a) The restoring force is proportional to the displacement x from equilibrium
b) The restoring force is proportional to x^3.

7. (**Statistical Mechanics**) Spin waves or magnons are elementary excitations in ferromagnetic materials. Like photons they are bosons, have zero rest mass and there are no conservation laws for the magnon number. Assume that at low temperatures the magnons obey a dispersion relation, $\epsilon = \hbar\omega = \hbar D k^2$, where D is a constant.
a) Calculate the density of states $g(\epsilon)$ for magnons.
b) Calculate the contribution of magnons to the specific heat at low temperatures. In this problem you need not explicitly evaluate the constants. Only the form (or power) of the temperature dependence is asked for.

8. (**Condensed Matter Physics**) Consider a dimerized linear chain where all the atoms are identical. As shown in the figure, M is the mass of the atoms and k_1, k_2 are the spring constants connecting two neighboring atoms. The spring constants are different so that $k_1 \neq k_2$. Calculate the sound velocity.



Optional (do 2 of of the final 4 problems)

9. (**Nuclear Physics**) The semi-empirical mass formula (SEMF) provides a good working model for the binding energy E_B of a nucleus and has the form

$$E_B(A, Z) = C_v A - C_s A^{2/3} - C_c \frac{Z^2}{A^{1/3}} - C_a \frac{(A - 2Z)^2}{A} ,$$

where A is the atomic weight and Z is the atomic number and the (C_v, C_s, C_c, C_a) are constants that are fit to nuclear data. (Note that we omit an asymmetry term).

a) For a fixed (constant) atomic weight A, find the optimized number N of neutrons (where $A = N + Z$). Express your result in terms of the ratio N/Z.

b) Using the result from the first part of the problem, derive an expression for the binding energy as a function of atomic weight A

c) Now consider a simplified version of the problem where all nuclei have equal numbers of protons and neutrons, so that $Z = N = A/2$. Derive an expression for the atomic weight A_* that has the highest binding energy per nucleon.

10. (**Condensed Matter Physics**) Many metals can occur with both the body-centered cubic and the face-centered cubic structure, and it is observed that the transition from one structure to the other involves only an insignificant volume change. Assuming no volume change, find the ratio D_{fcc}/D_{bcc} where D_{fcc} and D_{bcc} are the closest distances of approach of metal atoms for the two structures.

11. (**Particle Physics**) A unit of radiation dose is the rad, which corresponds to an energy deposit of 0.01 J/kg. The annual safe dose for humans is 5 rad. Let us assume that 100 times this dose would lead to the extinction of life. Based on this, find a lower limit on the proton lifetime. (Assume that in proton decay all of the liberated energy is absorbed in tissue.) The mass of the proton is 1.67×10^{-27} kg.

12. (**Atomic Physics**) The electron and positron have the same (absolute) magnetic moment, but opposite g-factors. Show that the "ground state" of the e^+-e^- atom (positronium), which is a $^1S_0$, $^3S_1$ doublet, cannot have a linear Zeeman effect. Argue in terms of the total magnetic-moment operator.