Name:_______________________________

Exam Number:_______

University of Michigan Physics Department Graduate Qualifying Examination

Part II: Modern Physics

Saturday 23 January 2016 9:30 am – 2:30 pm

This is a closed book exam, but a number of useful quantities and formulas are provided in the front of the exam. (**Note that this list is more extensive than in past years.**) If you need to make an assumption or estimate, indicate it clearly. Show your work in an organized manner to receive partial credit for it. Answer the questions directly in this exam booklet. If you need more space than there is under the problem, continue on the back of the page or on additional blank pages that the proctor will provide. Please clearly indicate if you continue your answer on another page. Label additional blank pages with your exam number, found at the upper right of this page (but not with your name). Also clearly state the problem number and "page x of y" (if there is more than one additional page for a given question).

You must answer the first 8 required questions and 2 of the 4 optional questions. Indicate which of the latter you wish us to grade (e.g. by circling the question number). We will only grade the indicated optional questions. Good luck!!

Some integrals and series expansions

$$\int_{-\infty}^{\infty} \exp(-\alpha x^2)\, dx = \sqrt{\frac{\pi}{\alpha}}$$

$$\int_{-\infty}^{\infty} x^2 \exp(-\alpha x^2)\, dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\exp(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots$$

Some Fundamental Constants

speed of light $c = 2.998 \times 10^{8}$ m/s

proton charge $e = 1.602 \times 10^{-19}$ C

Planck's constant $\hbar = 6.626 \times 10^{-34}$ J·s $= 4.136 \times 10^{-15}$ eV·s

Rydberg constant $R_{\infty} = 1.097 \times 10^{7}\,\mathrm{m}^{-1}$

Coulomb constant $k = (4\pi\varepsilon_0)^{-1} = 8.988 \times 10^{9}$ N·m^2 / C^2

vacuum permeability $\mu_0 = 4\pi \times 10^{-7}$ T·m/A

universal gas constant $R = 8.3$ J / K·mol

Avogadro′s number $\mathrm{N_A} = 6.02 \times 10^{23}\,\mathrm{mol}^{-1}$

Boltzmann′s constant $\mathrm{k_B} = \mathrm{R/N_A} = 1.38 \times 10^{-23}$ J/K $= 8.617 \times 10^{-5}$ eV/K

Stefan-Boltzmann constant $\sigma = 5.67 \times 10^{-8}\,\mathrm{W / m^2K^4}$

radius of the sun $R_{\mathrm{sun}} = 6.96 \times 10^{8}$ m

radius of the earth $R_{\mathrm{earth}} = 6.37 \times 10^{6}$ m

radius of the moon $R_{\mathrm{moon}} = 1.74 \times 10^{6}$ m

gravitational constant $G = 6.67 \times 10^{-11}\,\mathrm{m^3 / (kg \cdot s^2)}$

Required (do all of the first 8 problems)

1. (**Quantum Mechanics**) Consider one-dimensional scattering from a delta function potential $V(x) = -g\delta(x)$ where g is the strength of the potential. If we send in a particle from the left

$$\psi_{\text{inc}} = e^{ikx} \qquad x < 0 \text{ and } k > 0$$

there will be an amplitude for both reflection and transmission. The transmitted part may be written

$$\psi_{\text{trans}} = S_{21}e^{ikx} = \sqrt{T}e^{i(kx+\delta)} \qquad x > 0$$

where S_{21} is the scattering amplitude, T is the (real) transmission coefficient and δ is the phase shift.
a) Find the transmission coefficient T and phase shift δ as a function of k, and show that T is insensitive to the sign of g while δ changes sign if g changes sign.
b) Rewrite the scattering amplitude S_{21} as a function of the energy, and show that it has a pole at negative energy corresponding to the bound state of the delta function potential (assuming $g > 0$ so the potential is attractive).

2. (**Quantum Mechanics**) Consider a system of two spin-1/2 particles with total angular momentum given by $\vec{J} = \vec{S}_1 + \vec{S}_2$ and Hamiltonian

$$H = \omega_1 S_{1z} + \omega_2 S_{2z}$$

Initially (at time $t = 0$) the system is in a singlet state of total angular momentum. At time t, we measure the total angular momentum J^2. What results can be found, and with what probabilities?

3. (**Quantum Mechanics**) Consider a one dimensional quantum simple harmonic oscillator with an unperturbed Hamiltonian of the usual form

$$H_0 = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 \, .$$

Now consider a perturbation to the Hamiltonian of the form

$$H_1 = \lambda \left(\frac{m^2 \omega^3}{2\hbar} \right) x^4 \, ,$$

where λ is a dimensionless constant. Find the shift in the energy level of the ground state (for $\lambda \ll 1$).

4. (**Quantum Mechanics**) Consider a quantum particle of mass m moving in a Yukawa-type potential of the form

$$V = -V_0 \frac{a}{r} \mathrm{e}^{-r/a}\,.$$

Using the variational method and the trial wave function of the form

$$\psi = C\mathrm{e}^{-kr}\,,$$

find the following:
a) the normalization constant C
b) the trial energy function $E(k)$
c) the condition on the parameter k that optimizes energy
d) a constraint on the parameter a that allows for a bound state
NOTE: For condition for part c) can be left in implicit form; the constraint for part d) will be written in terms of a given value of k (which can be left arbitrary, but can be assumed to be the optimum value from part c).

5. (**Statistical Mechanics**) Consider the following type of random walk: A man is confined to a one-dimensional line and takes steps of length a. His motion depends on a series of indenpendent random events: For each trial, he throws a die. If the number is odd (1,3,5) then he takes that many steps to the left. If the number is even (2,4,6) then he takes that many steps to the right.

a) After N trials (N throws of the die), find the expectation value for the displacement $\langle X_N \rangle$.

b) After N trials, find the root-mean-square fluctuation Δ_N, where $\Delta_N^2 = \langle (X_N - \langle X_N \rangle)^2 \rangle$.

c) How many trials are required for the expected displacement to become greater than the RMS fluctuation?

6. (**Statistical Mechanics**) Suppose that a systems has N particles and the number of accessible states Ω is a function of the form $\Omega(N, U, V) = \Omega(N, X)$, where $X = UV^3$, and where U is the internal energy and V is the volume. In other words, the number of accessible states does not depend on U and V independently, but rather only depends on the combined quantity $X = UV^3$.
a) Suppose that the system initially has volume $V_0 = L_0^3$ and internal energy U_0. If the system undergoes isentropic expansion to a new volume $V_1 = L_1^3$ where $L_1 = 3L_0$, find an expression for the final energy U_1 (in terms of U_0).
b) Derive an expression for the pressure of this gas as a function of U and V.

7. (**Statistical Mechanics**) Consider a system with N spin-1 (distinguishable) particles. We will use N_+ (or N_-) to represent the number of particles with spin $S_z = +1$ (or $S_z = -1$). The number of particles with $S_z = 0$ is then $N - N_+ - N_-$.
a) Find the number of microscopic states for fixed N_+ and N_-.
b) Prove that the entropy of the system, as a function of N_+ and N_- is

$$S = k_B[N \ln N - N_+ \ln N_+ - N_- \ln N_- - (N - N_+ - N_-) \ln(N - N_+ - N_-)]$$

Here we assume that N, N_+, N_- and $N - N_+ - N_-$ are all large enough, so that we can use the following approximation $\ln n! = n \ln n - n$.
c) Prove that the entropy, as a function of N_+ and N_-, is maximized at $N_+ = N_- = N - N_+ - N_- = N/3$.

8. (**Condensed Matter Physics**) In a 3D solid, there exist both longitudinal and transverse sound waves.

a) How many independent polarizations do the longitudinal acoustic modes have? How about transverse acoustic modes?

b) What is the energy of a phonon for a sound wave at frequency ω?

c) For a fixed frequency ω, the probability for having n phonons is proportional to $e^{-\beta E}$, where E is the total energy of the n phonons. What is the average number of phonons for a fixed frequency ω? Show that it recovers the Bose-Einstein statistics. [Hint: you may use the following equations $\sum_n ne^{-nx} = -\frac{d}{dx}\sum_n e^{-nx}$ and $\sum_{n=0}^{\infty} e^{-nx} = 1/(1-e^{-x})$.]

Optional (do 2 of of the final 4 problems)

9. (**Nuclear Physics**) The nucleus ^{57}Fe has an excited state with energy 14.4keV that decays with a half-life of 10^7 sec by the emission of a gamma ray photon.

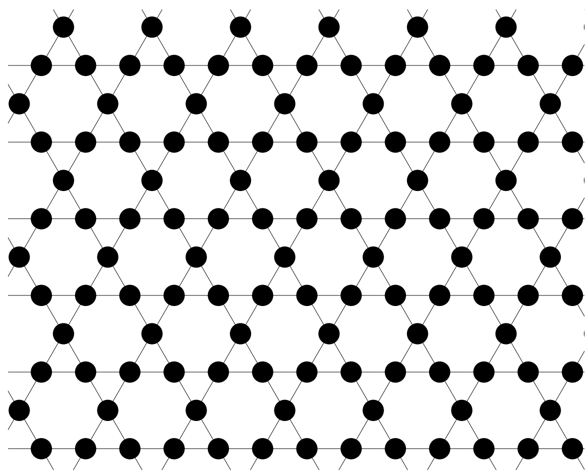
a) What is the width of the gamma ray line?

b) What is the recoil energy of the free ^{57}Fe nucleus?

c) Explain why the emitted gamma ray photon cannot be absorbed by a second free ^{57}Fe nucleus at rest.

d) Find the minimum velocity that the second nucleus must have in order to absorb the gamma ray photon.

10. (**Condensed Matter Physics**) In the 2D lattice shown below, the length for each bond is a and the angles between to neighboring bonds are either $60°$ or $120°$.
a) How many sites are there in a unit cell?
b) Find the two primitive lattice vectors $\vec{a}_1$ and $\vec{a}_2$ (write down the x and y components for each vector. Here the horizontal and vertical directions are the x and y directions respectively).
(c) Find the reciprocal lattice vectors (write down the x and y components for each vector).



11. (**Particle Physics**) At an accelerator, a beam pulse of 100 GeV protons contains 10^{10} such protons focused onto a spot of area 2 cm^2, and extracted uniformly over a time of 0.5 sec. This pulse is directed onto a copper target of thickness 0.1cm, where nuclear interactions are expected to occur with a total collision cross-section of 1 mb (10^{-27} cm^2). If a detector could be built to detect every scattered proton, what would be the counting rate in that detector when the beam pulse was being delivered? Assume that copper has density $\rho = 8.96$ g/cm^3 and atomic mass number $A = 63.5$.

12. (**Atomic Physics**) In a hydrogen-like atom, the $2S$ and $2P$ levels are separated by a small energy difference, Δ, due to a small effect which has a negligible influence on the wave functions of these states. The atom is placed in an electric field of magnitude E. Neglecting electron spin and the influence of more distant levels, obtain a general expression for the energy shifts of the $n = 2$ levels as a function of the field strength E. Note: Do not evaluate explicitly any nonzero integrals which may occur in your discussion.