Name:________________________________

Exam Number:_______

University of Michigan Physics Department
Graduate Qualifying Examination

Part II: Modern Physics
Saturday 18 January 2014 9:30 am – 2:30 pm

This is a closed book exam, but a number of useful quantities and formulas are provided in the front of the exam. (**Note that this list is more extensive than in past years.**) If you need to make an assumption or estimate, indicate it clearly. Show your work in an organized manner to receive partial credit for it. Answer the questions directly in this exam booklet. If you need more space than there is under the problem, continue on the back of the page or on additional blank pages that the proctor will provide. <u>Please clearly indicate if you continue your answer on another page. Label additional blank pages with your exam number, found at the upper right of this page (but not with your name). Also clearly state the problem number and "page x of y" (if there is more than one additional page for a given question).</u>

You must answer the first 8 required questions and 2 of the 4 optional questions. Indicate which of the latter you wish us to grade (e.g. by circling the question number). We will only grade the indicated optional questions. Good luck!!

Some integrals and series expansions

$$\int_{-\infty}^{\infty} \exp(-\alpha x^2)\,\mathrm{d}x = \sqrt{\frac{\pi}{\alpha}}$$

$$\int_{-\infty}^{\infty} x^2 \exp(-\alpha x^2)\,\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\exp(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots$$

<u>*Some Fundamental Constants*</u>

speed of light $c = 2.998\times10^{8}$ m/s

proton charge $e = 1.602\times10^{-19}$ C

Planck's constant $\mathrm{h} = 6.626\times10^{-34}$ J·s $= 4.136\times10^{-15}$ eV·s

Rydberg constant $R_\infty = 1.097\times10^{7}\,\mathrm{m}^{-1}$

Coulomb constant $k = (4\pi\varepsilon_0)^{-1} = 8.988\times10^{9}$ $\mathrm{N{\cdot}m^2/C^2}$

vacuum permeability $\mu_0 = 4\pi\times10^{-7}$ T·m/A

universal gas constant $R = 8.3$ J / K·mol

Avogadro′s number $\mathrm{N_A} = 6.02\times10^{23}\,\mathrm{mol^{-1}}$

Boltzmann′s constant $\mathrm{k_B = R/N_A} = 1.38\times10^{-23}$ J/K $= 8.617\times10^{-5}$ eV/K

Stefan-Boltzmann constant $\sigma = 5.67\times10^{-8}\,\mathrm{W/m^2K^4}$

radius of the sun $R_{\mathrm{sun}} = 6.96\times10^{8}$ m

radius of the earth $R_{\mathrm{earth}} = 6.37\times10^{6}$ m

radius of the moon $R_{\mathrm{moon}} = 1.74\times10^{6}$ m

gravitational constant $G = 6.67\times10^{-11}\,\mathrm{m^3/(kg{\cdot}s^2)}$

REQUIRED: DO ALL OF PROBLEMS 1 – 8

1. (Quantum Mechanics) A square well potential in one dimension is given by

$$V(x) = \begin{cases} \infty & x < 0 \\ 0 & 0 \leq x \leq a \\ V_0 & x > a \end{cases},$$

where V_0 and a are positive constants. Derive the condition that guarantees that there is at least one bound state eigenfunction.

2. (Quantum Mechanics) Consider a particle of charge q in a simple harmonic oscillator potential with Hamiltonian

$$H_0 = \frac{p^2}{2m} + \frac{1}{2}m^2\omega^2 x^2.$$

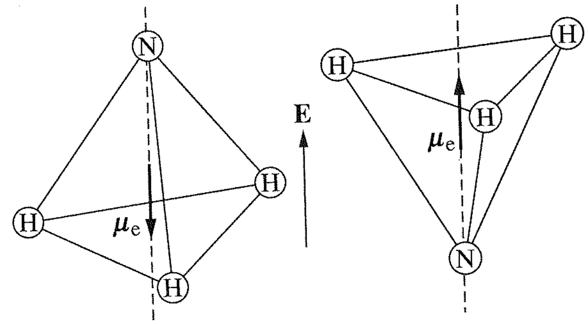
Now add a perturbation froma constant electric field

$$H_1 = -q|E|x$$

(a) What is the first-order perturbation to the energy eigenvalues for a state $|n\rangle$?

(b) Using second order perturbation theory or otherwise, what is the shift in energies proportional to $|E|^2$?

3. (Quantum Mechanics) Consider two states of ammonia, distinguished by the relative position of the N atom relative to the three Hydrogen atoms, see Figure. Denote the state in (a) as $|1\rangle$ and the state in (b) as $|2\rangle$.



(a) These states are not eigenvalues of the hamiltonian. Rather, the Hamiltonian is given by

$$H = E_0|1\rangle\langle 1| + (-A)|1\rangle\langle 2| + (-A)|2\rangle\langle 1| + E_0|2\rangle\langle 2|$$

What are the eigenvalues and eigenstates of the Hamiltonian?

(b) Now suppose the atom is placed in an external electric field as shown in the Figure. The location of the N atom causes the $|1\rangle$ and $|2\rangle$ states to have differing electric dipole moments. These states *are* eigenstates of the interaction

$$H_{int} = -\mu_e \cdot E$$

Find the new eigenvalues of the Hamiltonian.

(c) Now suppose that the energy shift induced by the interaction with electric field is small with respect to A. In this case, write an approximate value for the energy eigenvalues.

(d) Finally, suppose a beam of ammonia molecules with velocity v is sent into a region of length ℓ where the electric field is inhomogeneous with a small constant gradient $\delta E \equiv \partial|E|/\partial z$ perpendicular to the direction of motion. In terms of the ammonia mass, m_{NH3}, to leading non-vanishing order in δE and μ_e, what is the separation in the beam components in z by the end of the region of length ℓ?

4. (Stat Mech) Imagine that you live in the 2-dimensional world Flatland and that you are the proud owner of a square box of side L filled with electromagnetic radiation at temperature T. (Assume that photons in 2-d can have only a single polarization.)

   (a) What is the average number $\langle s_{\mathbf{k}} \rangle$ of photons of wavevector $\mathbf{k}$ present in your box?

   (b) What is the spectral energy density u_ω—that is, the average energy per volume per unit frequency in the box?

   (c) Thus, what form does the Stefan-Boltzmann law take in 2 dimensions? You need not give numerical prefactors, but be sure to show the correct dependence on all parameters and fundamental constants. (Recall that the Stefan-Boltzmann law gives the total power radiated per unit surface area by a black body at temperature T, summed over all frequencies.)

5. (Stat Mech) A diatomic molecule can be idealized as two point particles joined by a stiff spring. The system thus has both rotational degrees of freedom (associated with the rotation of the entire assembly as a rigid body) and vibrational degrees of freedom (associated with relative motions of the two atoms). More specifically, the states of such a molecule are specified by a rotational quantum number j and a vibrational quantum number n (both non-negative integers). They have energies

$$\mathcal{E}_{j,n} = \frac{\hbar^2 j(j+1)}{2I} + \hbar\omega_0 n \; ,$$

where I is the molecule's moment of inertia and ω_0 is the natural vibrational frequency of the bond joining the two molecules, and multiplicities

$$g(j,n) = 2j+1 \; .$$

   (a) Write down the partition function of such diatomic molecule. (Simply write it down, do not attempt to perform any sums at this point.)

   (b) The figure below shows a sketch of the specific heat of a collection of hydrogen molecules as a function of temperature. Taking into consideration that $C = \tau d\sigma/d\tau$, give a qualitative explanation for the behavior of the specific heat presented below based on general properties of the entropy.

   (c) Identify the two distinct energy scales in the problem and find approximate expressions for the value of T_1 and T_2 indicated in the figure below.

6. (Atomic Physics) Use simple physical arguments to estimate the ratio of the lowest lying electronic, vibrational, and rotational transition frequencies of a homonuclear diatomic molecule. Your answer should depend only on the ratio of electron to nuclear mass. Using your results and your knowledge of the frequency of electronic transitions, determine whether or not rotational, vibrational and electronic transitions contribute to the specific heat of a gas of diatomic molecules at room temperature.

7. (Particle Physics) The Higgs boson was discovered last year at the LHC, primarily based on the evidence from two decay channels: $h \to \gamma\gamma$ and $h \to ZZ^* \to 4$ leptons. Here Z^* represents an "off-shell" Z: one can think of these $h \to ZZ^*$ decays as decays to a Z boson and a pair of leptons (the products of the off-shell Z's decay). The Higgs boson weighs about 125 GeV/c^2.

   (a) Consider an event with 4 leptons. In this event, a pair of muons reconstructs to the Z-boson mass (91 GeV/c^2). There is a second pair of electrons in the event. Assuming this event arises from a Higgs boson decay, what is the maximum invariant mass of the lepton pair?

   (b) At the observed mass, the most common decay of the Higgs boson is to a pair of b quarks, why wasn't this decay used in the discovery of the Higgs boson?

   (c) It is often said that the Higgs boson couples to the mass of the particle. Photons, however, are massless. How is it possible to have discovered the Higgs via the decay to photons? Draw a Feynman diagram that mediates the decay of the Higgs boson to photons.

   (d) The cross section for Higgs bosons production at the LHC running at 8 TeV is about 20 pb. The Standard Model branching ratio to photons is about 2×10^{-3} . In 20 fb^{-1} of data (the amount taken by the end of the last run), how many Higgs boson decays to photons were expected? Assume that the detection efficiency was 50%.

8. (Condensed Matter) Consider a partially filled band in a one-dimensional solid. The dispersion relation is

$$\varepsilon(k) = -2t\cos(ka)$$

Here, $t > 0$ is a control parameter. a is the lattice constant and k is the wavevector. For simplicity, we set the Planck constant $\hbar = 1$ and we ignore all other bands as well as interactions between electrons. We also assume that the Fermi energy E_F satisfies $-2t < E_F < 2t$ (i.e., the band is partially filled).

   (a) Find the Fermi wavevector, the Fermi velocity and the effective mass in terms of t, a and E_F.

   (b) Find the density of state at the Fermi energy in terms of t, a and E_F.

   (c) Find the total energy of the system at $T = 0$ in terms of t, a, E_F and the system size L.

OPTIONAL: DO 2 OF THE FOLLOWING 4 PROBLEMS

9. (Quantum Mechanics) Ignoring the effects of turning points, obtain the formula for the WKB approximation to the wave function:

$$\psi(x) \approx \frac{C}{\sqrt{p(x)}} e^{\pm \frac{i}{\hbar} \int p(x)dx}. \tag{1}$$

   (a) Start with the assumption that the form for the wave function is

$$\psi(x) = A(x) e^{i\phi(x)}, \tag{2}$$

   and derive expression (1).

   (b) Highlight the main approximation/simplification in the derivation above. Give a physical explanation for this approximation.

   (c) How is this function related to the classical probability density of finding a particle in the interval $(x, x+dx)$

10. (Atomic Physics)

(a) What is the physical origin of the fine-structure splitting? Using a simple physical model for the electron in the $n = 2$ state of hydrogen, estimate the spin-orbit splitting of the $2P$ states in GHz?

(b) What is the physical origin of the hyperfine splitting? Explain why there is no fine structure splitting in the ground state of hydrogen, but that levels are split by the hyperfine interaction (you are not asked to calculate this splitting). Indicate the total angular momenta of the split levels and the degeneracy of each of these states. The cosmological 21 cm line results from a transition between the ground state hyperfine levels. Based on this fact, what is the hyperfine separation of the ground state hypefine levels? Is the radiation electric dipole or magnetic dipole? Explain.

(c) Carbon has 6 electrons. Write the electron configuration for the ground state of carbon. Neglecting the Pauli Principle, what are the possible ground state levels of carbon in the Russell-Saunders (LS) coupling scheme. Taking into account the Pauli Principle, which of these states must be excluded? Explain your reasoning. The Hund's rules state that:

1. For a given electron configuration, the term with maximum total spin S has the lowest energy.

2. Within a state of fixed total spin S, the state having the largest L has the lowest energy.

3. In an atom with outermost subshell half-filled or less, the level with the lowest value of the total angular momentum quantum number J lies lowest in energy.

Armed with this fact, list the ground state levels in order of increasing energy, expressed in the form ${}^{2S+1}L_J$.

11. (Condensed Matter) The Ginzburg-Landau free energy F of a superconductor is given by

$$F = \int d\vec{r} \left\{ \alpha \left[-i\hbar\nabla - 2e\vec{A}(\vec{r}) \right]^* \Psi^*(\vec{r}) \left[-i\hbar\nabla - 2e\vec{A}(\vec{r}) \right] \Psi(\vec{r}) + \beta\Psi(\vec{r})^*\Psi(\vec{r}) + \gamma\Psi(\vec{r})^*\Psi(\vec{r})^*\Psi(\vec{r})\Psi(\vec{r}) \right\},$$

where α, β and γ are control parameters. For simplicity, we assume $\alpha > 0$, $\beta < 0$ and $\gamma > 0$. $\Psi(\vec{r})$ is the complex order parameter and * indicates the complex conjugate. The first term in the Ginzburg-Landau free energy comes from the kinetic energy, which is proportional to the square of the momentum (P^2). Here, because a Cooper pair carries charge $2e$, the momentum operator is $\vec{P} = -i\hbar\nabla - 2e\vec{A}(\vec{r})$, instead of $-i\hbar\nabla$, where $\vec{A}(\vec{r})$ is the vector potential.

(a) Let us rewrite the order parameter as $\Psi(\vec{r}) = \rho \exp[i\phi(\vec{r})]$. For simplicity, we assume that ρ is a constant independent of $\vec{r}$, while the phase $\phi(\vec{r})$ is a function of $\vec{r}$. Rewrite the Ginzburg-Landau free energy in terms of ρ and $\phi(\vec{r})$.

(b) Determine (a) the value of ρ in terms of β and γ and (b) the relation between $\phi(\vec{r})$ and $\vec{A}$ by minimizing the Ginzburg-Landau free energy (Hint: $\alpha > 0$, $\beta < 0$ and $\gamma > 0$).

(c) Prove that the magnetic field inside a superconductor is zero (i.e. the Meissner effect) using the results of part (2) (Hint: $B = \nabla \times A$).

12. (Nuclear Physics) The Sun converts protons into Helium through a series of reactions that has the effective form

$$4p \to {}^4\mathrm{He} + 2e^+ + 2\nu_e\,.$$

The solar constant describing the power of the solar radiation at the location of Earth is S = 1400 W/m^2. The energy released per reaction corresponds to the binding energy of He (28.3 MeV).

(a) Find the flux of neutrinos arriving at Earth, i.e, the number of neutrinos that hit the Earth's surface in units of number per square meter per second.

(b) Estimate the ratio of the power of the neutrino flux to the power of the solar photon flux. Assume that the average energy of a neutrino is 0.3 MeV.