Name:________________________________

Exam Number:_______

University of Michigan Physics Department Graduate Qualifying Examination

Part I: Classical Physics
Saturday 16 January 2016 9:30 am – 2:30 pm

This is a closed book exam, but a number of useful quantities and formulas are provided in the front of the exam. (**Note that this list is more extensive than in past years.**) If you need to make an assumption or estimate, indicate it clearly. Show your work in an organized manner to receive partial credit for it. Answer the questions directly in this exam booklet. If you need more space than there is under the problem, continue on the back of the page or on additional blank pages that the proctor will provide. <u>Please clearly indicate if you continue your answer on another page. Label additional blank pages with your exam number, found at the upper right of this page (but not with your name). Also clearly state the problem number and "page x of y" (if there is more than one additional page for a given question).</u>

You must answer the first 8 required questions and 2 of the 4 optional questions. Indicate which of the latter you wish us to grade (e.g. by circling the question number). We will only grade the indicated optional questions. Good luck!!

Some integrals and series expansions

$$\int_{-\infty}^{\infty} \exp(-\alpha x^2)\,\mathrm{d}x = \sqrt{\frac{\pi}{\alpha}}$$

$$\int_{-\infty}^{\infty} x^2 \exp(-\alpha x^2)\,\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\exp(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots$$

Some Fundamental Constants

speed of light $c = 2.998 \times 10^{8}$ m/s

proton charge $e = 1.602 \times 10^{-19}$ C

Planck's constant $\hbar = 6.626 \times 10^{-34}$ J·s $= 4.136 \times 10^{-15}$ eV·s

Rydberg constant $R_{\infty} = 1.097 \times 10^{7}\,\mathrm{m}^{-1}$

Coulomb constant $k = (4\pi\varepsilon_0)^{-1} = 8.988 \times 10^{9}$ N·m^2 / C^2

vacuum permeability $\mu_0 = 4\pi \times 10^{-7}$ T·m/A

universal gas constant $R = 8.3$ J / K·mol

Avogadro′s number $\mathrm{N_A} = 6.02 \times 10^{23}\,\mathrm{mol}^{-1}$

Boltzmann′s constant $\mathrm{k_B} = \mathrm{R/N_A} = 1.38 \times 10^{-23}$ J/K $= 8.617 \times 10^{-5}$ eV/K

Stefan-Boltzmann constant $\sigma = 5.67 \times 10^{-8}\,\mathrm{W / m^2K^4}$

radius of the sun $R_{\mathrm{sun}} = 6.96 \times 10^{8}$ m

radius of the earth $R_{\mathrm{earth}} = 6.37 \times 10^{6}$ m

radius of the moon $R_{\mathrm{moon}} = 1.74 \times 10^{6}$ m

gravitational constant $G = 6.67 \times 10^{-11}\,\mathrm{m^3 / (kg{\cdot}s^2)}$

Required (do all of the first 8 problems)

1. (**Mechanics**) A pendulum is made from a massless spring (with force constant k and unstretched length ℓ_0) that is suspended at one end from a fixed pivot O and has a mass m attached to its other end. The spring can stretch and compress but cannot bend, and the whole system is confined to a single vertical plane.

(a) Write down the Lagrangian $\mathcal{L}$ for the pendulum, using as generalized coordinates the usual angle ϕ and the length r of the spring.
(b) Derive the Euler-Lagrange equations from the Lagrangian $\mathcal{L}$.
(c) Find the solutions for small oscillations. [Hint: Let ℓ denote the equilibrium length of the spring with the mass hanging from it and write $r = \ell + \epsilon$.]

2. (**Mechanics**) A bead of mass m slides without friction (but under the influence of gravity) on a wire that is bent in the shape of a curve described by $z = f(\rho)$ where $f(\rho)$ is a function to be determined. The wire is spun with constant angular velocity ω about the vertical axis (i.e., the $\hat{z}$ axis). Here we are using cylindrical coordinates (ρ, ϕ, z). Find the function $f(\rho)$ (i.e., the shape of the wire) so that the bead can be at equilibrium at *any* fixed value of ρ. You do not have to check the stability of this equilibrium.

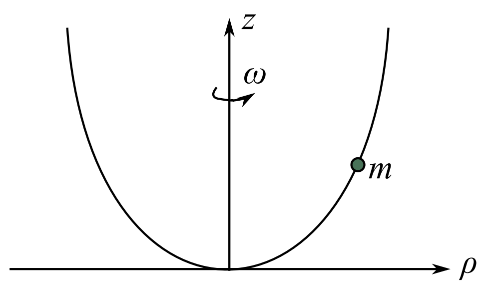
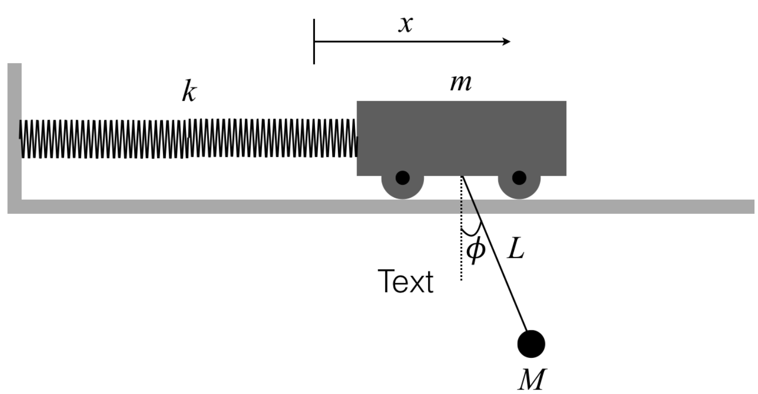


Figure 1: A wire whose shape is described by the curve $z = f(\rho)$ is spun about the $\hat{z}$ axis.

3. (**Mechanics**) A simple pendulum (mass M and length L) is suspended from a cart (mass m) that can oscillate on the end of a spring of force constant k, as shown in the figure below.

(a) Assuming that the angle ϕ remains small, write down the Lagrangian $\mathcal{L}$ for the system.
(b) Derive the Euler-Lagrange equations from the Lagrangian $\mathcal{L}$.
(c) Assuming that $m = 1$, $M = 1$, $L = 1$, g $= 1$, and $k = 2$ (all in appropriate units), find the normal frequencies and the corresponding normal modes for the system.



4. (**Thermodynamics**) If we raise the temperature from T to $T + \delta T$ (through a quasi-static process), the entropy of a system increases by

$$dS = \frac{\delta Q}{T} = \frac{C(T)}{T} dT \tag{1}$$

where $C(T)$ is the heat capacity at temperature T. If we define the entropy at $T = 0$ to be zero, using the formula above, we find that

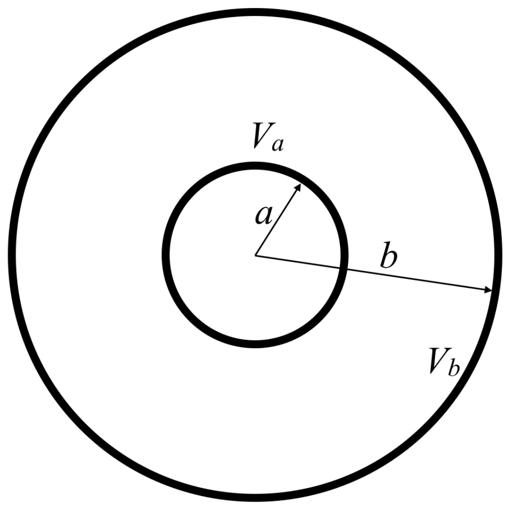
$$S(T) = \int_0^T dS = \int_0^T \frac{C(T')}{T'} dT' \tag{2}$$

a) Prove that the heat capacity must vanish at $T = 0$, i.e. $\lim_{T\to 0} C(T) = 0$. [Hint: Prove this conclusion by showing that the integral above diverges, if $\lim_{T\to 0} C(T)$ is nonzero.]

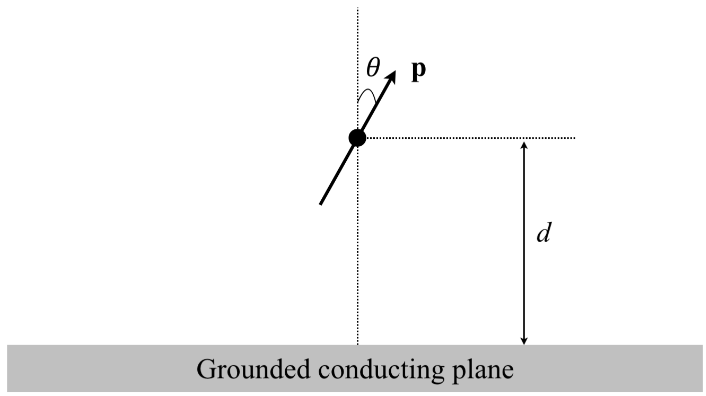
b) As proved in part a), $\lim_{T\to 0} C(T) = 0$ is a necessary condition for the entropy (defined using the integral above) to converge. However, is this a sufficient condition? Answer this question by considering $C(T) = 1/\ln T$. As $T \to 0$, this $C(T)$ approaches zero. But does the integral for entropy converge using this $C(T)$? [Hint: The integral can be evaluated by defining $x = \ln T'$ and transferring $\int dT'$ into $\int dx$.]

5. (**E&M**) Two very thin concentric spherical conducting shells are separated by vacuum as shown in the figure below. The inner shell with radius a is held at potential V_a, while the outer shell with radius b is held at potential V_b.

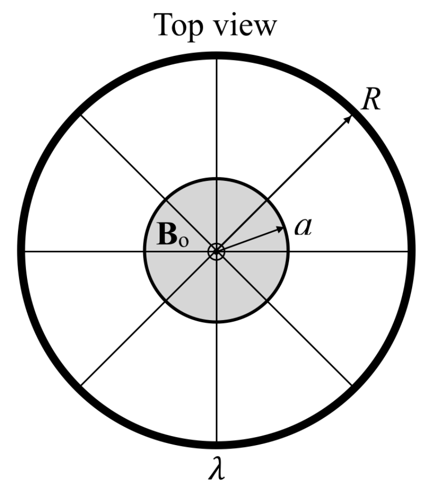
(a) Calculate the potential $\phi(r)$ between the inner and outer shells and express the solution in terms of a, b, V_a and V_b.
(b) Calculate the charge q_a on the outer surface of the inner conducting spherical shell and the charge q_b on the inner surface of the outer conducting spherical shell.
(c) Calculate the coefficients of capacitance C_{ij} and express it in terms of the radii a and b of the two concentric spherical shells. [Hint: $q_i = \sum_j C_{ij} V_j$.]



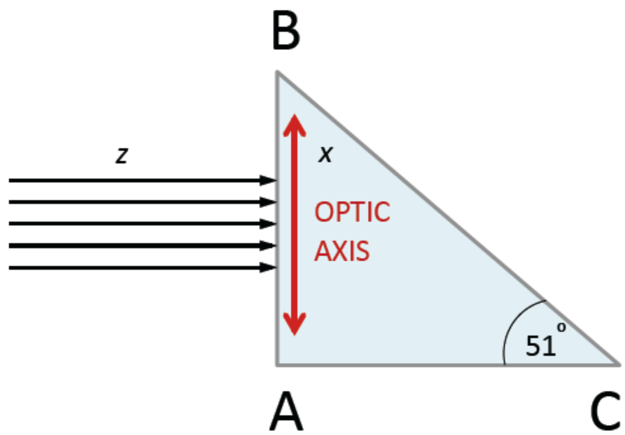
6. (**E&M**) A dipole $\mathbf{p}$ is situated a distance d above an infinite grounded conducting plane as shown in the figure below. The dipole makes an agnle θ with the perpendicular to the plane. Find the torque on the dipole $\mathbf{p}$.



7. (**E&M**) A line charge λ is glued onto the rim of a wheel of radius R, which is then suspended horizontally as shown in the figure below, so that it is free to rotate (the spokes are made of nonconducting material). In the central region, out to radius a, there is a uniform magnetic field $\mathbf{B_o}$, pointing up. If the magnetic field is now switched off, calculate the total angular momentum imparted to the wheel.



8. (**Optics**) A beam of monochromatic ($\lambda = 0.5893\ \mu$m) and linearly polarized light moving along the $\hat{z}$ axis, reaches normally the surface AB of a prism of calcite ($n_0 = 1.6584$ and n_e =1.4864). The electric field subtends an angle θ with the plane of incidence. The optical axis is along the $\hat{x}$ axis, parallel to the surface AB (see figure). Find what happens to the beam through the prism and when it emerges from the surface BC.



Optional (do 2 of of the final 4 problems)

9. (**Mechanics**) The Coriolis force can produce a torque on a spinning object. To illustrate this, consider a horizontal hoop of mass m and radius r spinning with angular velocity ω about its vertical axis at colatitude θ. Calculate the torque on the spinning horizontal hoop produced by the Coriolis force due to the earth's rotation of angular velocity $\mathbf{\Omega}$. This torque is the basis of the gyrocompass. [Hint: Let S be a non-inertial frame rotating with constant angular velocity $\mathbf{\Omega}$ relative to the inertial frame S', and let the origin of the non-inertial frame coincide with that of the inertial frame. Note: $\mathbf{v}_{S'} \equiv (d\mathbf{r}/dt)_{S'} = (d\mathbf{r}/dt)_S + \mathbf{\Omega} \times \mathbf{r} = \dot{\mathbf{r}} + \mathbf{\Omega} \times \mathbf{r}$, $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$, and $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$.]

10. (**E&M**) A spherical linear dielectric medium of radius a and dielectric constant ϵ/ϵ_0 is placed at the origin of free space. A point electric dipole with dipole moment $\vec{p}$ is then placed at the origin (centered inside the dielectric).
a) An observer outside the dielectric sphere will see a screened electric field. Show that this field is a pure dipole field.
b) What is the value of the screened dipole moment seen by the outside observer?

11. (**Thermodynamics**) In a room at temperature 27° C, Bob uses a heat pump to cook a dish (say, the main course for a meal) at a temperature of 127° C. The heat pump gathers the heat from the air and transfers it to the dish. For every Joule of energy that the heat pump consumes, how much heat can the dish obtain in the ideal limit?

12. (**Optics**) A lens of refractive index $n_2 = 1.85$ is covered on the top surface with a very thin layer of a transparent substance, whose refractive index is n_1 and thickness is t. A beam of monochromatic light ($\lambda = 0.4$ μm) is incident normally on the layer and the lens (see figure). Assuming that the reflectivities are the same for the air-layer and layer-lens boundary surfaces, find the values n_1 and t for which the reflected ray is subject to destructive interference.