Name:________________________________

Exam Number:________

University of Michigan Physics Department
Graduate Qualifying Examination

Part II: Modern Physics

Saturday, May 18, 2013 9:30 am – 2:30 pm

This is a closed book exam, but a number of useful quantities and formulas are provided in the front of the exam. (**Note that this list is more extensive than in past years.**) If you need to make an assumption or estimate, indicate it clearly. Show your work in an organized manner to receive partial credit for it. Answer the questions directly in this exam booklet. If you need more space than there is under the problem, continue on the back of the page or on additional blank pages that the proctor will provide. Please clearly indicate if you continue your answer on another page. Label additional blank pages with your exam number, found at the upper right of this page (but not with your name). Also clearly state the problem number and "page x of y" (if there is more than one additional page for a given question).

You must answer the first 8 required questions and 2 of the 4 optional questions. Indicate which of the latter you wish us to grade (e.g. by circling the question number). We will only grade the indicated optional questions. Good luck!!

Some integrals and series expansions

$$\int_{-\infty}^{\infty} \exp(-\alpha x^2)\, \mathrm{d}x = \sqrt{\frac{\pi}{\alpha}}$$

$$\int_{-\infty}^{\infty} x^2 \exp(-\alpha x^2)\, \mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\exp(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$(1+x)^{\alpha} = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots$$

Some Fundamental Constants

speed of light $c = 2.998 \times 10^{8}$ m/s

proton charge $e = 1.602 \times 10^{-19}$ C

proton mass $m_p = 1.67 \times 10^{-27}$ kg

electron mass $m_e = 9.11 \times 10^{-31}$ kg

Planck's constant $\hbar = 6.626 \times 10^{-34}$ J·s $= 4.136 \times 10^{-15}$ eV·s

Rydberg constant $R_\infty = 1.097 \times 10^{7}\,\mathrm{m}^{-1}$

Coulomb constant $k = (4\pi\epsilon_0)^{-1} = 8.988 \times 10^{9}\ \mathrm{N{\cdot}m^2 / C^2}$

vacuum permeability $\mu_0 = 4\pi \times 10^{-7}\,\mathrm{T{\cdot}m/A}$

universal gas constant $R = 8.3\ \mathrm{J / K{\cdot}mol}$

Avogadro′s number $\mathrm{N_A} = 6.02 \times 10^{23}\,\mathrm{mol^{-1}}$

Boltzmann′s constant $\mathrm{k_B} = \mathrm{R/N_A} = 1.38 \times 10^{-23}\,\mathrm{J/K} = 8.617 \times 10^{-5}\,\mathrm{eV/K}$

Stefan-Boltzmann constant $\sigma = 5.67 \times 10^{-8}\,\mathrm{W / m^2K^4}$

radius of the sun $R_{\mathrm{sun}} = 6.96 \times 10^{8}\,\mathrm{m}$

radius of the earth $R_{\mathrm{earth}} = 6.37 \times 10^{6}\,\mathrm{m}$

mass of the sun $M_{\mathrm{sun}} = 1.99 \times 10^{-30}$ kg

mass of the earth $M_{\mathrm{earth}} = 5.97 \times 10^{24}\,\mathrm{kg}$

gravitational constant $G = 6.67 \times 10^{-11}\,\mathrm{m^3 / (kg{\cdot}s^2)}$

Required (do all problems)

1. (**Quantum Mechanics**) Consider a particle in a one-dimensional potential well centered on $x = 0$. In the limit where the potential is narrow and deep, we can model it as a Dirac delta function

$$V(x) = -V_0 l \delta(x),$$

where V_0 is a constant (with units of energy) that determines the depth of the well and l is a length scale.

a) Find the energy levels of the bound states of this system.
b) Suppose we add a small perturbation to $V(x)$ of the form

$$U(x) = U_0 \left| \frac{x}{a} \right|.$$

Find the shift in the ground state energy induced by this perturbation.

2. (**Quantum Mechanics**) Answer the following two questions:

(a) A hydrogen atom is placed in a homogenous static electric field $\vec{\epsilon} = \epsilon\hat{\mathbf{z}}$. The Hamiltonian may be written as $H = H_0 - \mu_z \epsilon$, where H_0 represents the Hamiltonian in the absence of the electric field and μ_z is the *z* component of the electric dipole moment operator. Show that the induced electric dipole moment in state *n* is:

$$\langle \mu_z \rangle_n = \langle n | \mu_z | n \rangle = -\frac{\partial E_n}{\partial \epsilon}.$$

You should <u>not</u> assume that ϵ is small or anything similar in your proof (but you may ignore the possibility of level crossings).

(b) Consider the following Hamiltonian for a 2-state system:

$$H = a(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1|)$$

Derive the matrix representation of H in the basis of |1> and |2> and determine the energy eigenvalues.

3. (**Quantum Mechanics**) Consider a particle placed in a one-dimensional infinite square well potential such that $V(x) = 0$ for $0 < x < a$, but $V(x) = \infty$ everywhere else.

a) Find the wavefunction and energy for both the ground state and the first excited state.

Now, consider two (non-interacting) particles placed in this potential. What are the wavefunctions and energies of the ground and first excited states of this two particle system in the cases where

b) the two particles are distinguishable (i.e., different species of particles altogether),
c) the two particles are identical bosons, and,
d) the two particles are identical fermions?

4. (**Quantum Mechanics**) A particle (in 3 dimensions) is placed in an infinite spherical potential where $V(r)=0$ for $0<r<a$, but $V(r)=\infty$ elsewhere. Derive the radially dependent part of the wavefunction and the energy eigenvalues of all states with zero angular momentum. (Do not worry about normalizing the wave function properly. You may find it useful to know that

$$\nabla^2 f = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rf) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

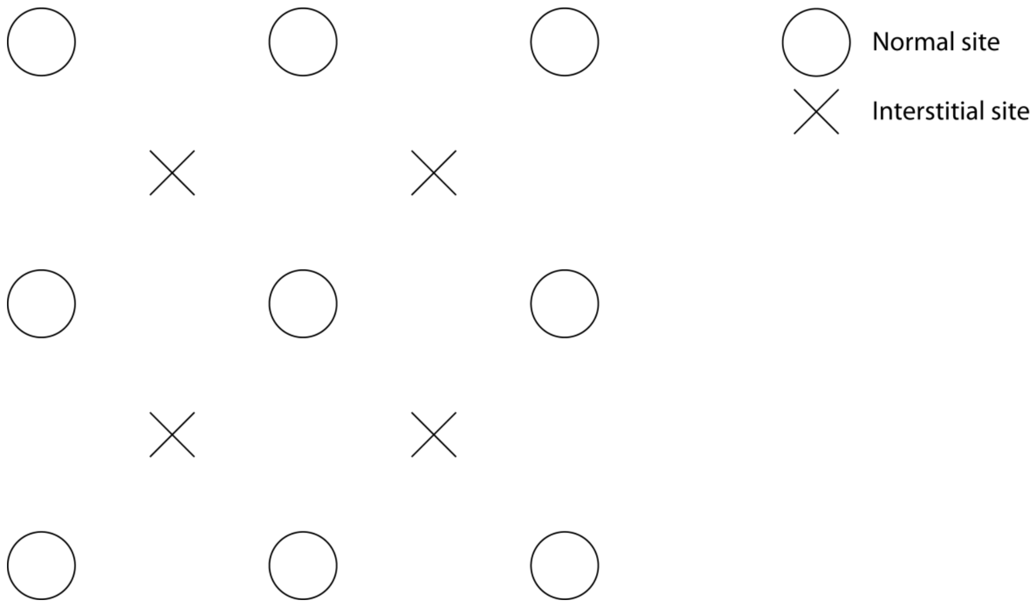
in spherical coordinates.)

5. (**Statistical Mechanics**) Consider a gas of non-interacting particles in 3 dimensions. Assume that these particles follow the classical Boltzmann distribution and that the energy of a particle is

$$E = \frac{p_x^2 + p_y^2 + p_z^2}{2m} + ax^2 + bx + c,$$

where a, b, and c are positive constants and the particle's position is (x,y,z). Find the average energy per particle at temperature T.

6. (**Statistical Mechanics**) Consider a two-dimensional crystal, as shown. Usually, atoms sit at the "normal" sites, indicated by circles, where they have the lowest energy. At non-zero temperature, however, it is also possible for some atoms to be excited into the "interstitial" sites, indicated by X's. An atom in an interstitial site has energy $\varepsilon_0 > 0$, while one in a normal site has energy 0.



a) Suppose that the N atoms are divided between a total of N normal sites and N interstitial sites. If n of these atoms are in interstitial sites, what are the system's internal energy U and entropy S? Assume the locations of the filled interstitial and empty normal sites are uncorrelated and that you are in the thermodynamic limit $N \gg 1$ and $n \gg 1$. *Hint*: If there are n filled interstitial sites, there must also be n empty normal sites, and there is entropy associated with both.
b) Find the fraction n / N of atoms that are excited into interstitial sites as a function of the temperature T.

7. (**Statistical Mechanics**) Consider a 2D free (non-interacting) Fermi gas. The gas is composed of spin ½ particles with energy $\epsilon = p^2/2m$. [*Note*: Part b) can be done by taking a limit of the result in part a) or by directly studying the ground state of the Fermi gas; thus, try b) even if you can't get a).]

a) Find the chemical potential μ as a function of the temperature T and the particle number density n. [*Hint:* Find n as a function of μ, then invert the relation. You may find it useful to know that $\int_0^\infty dx \frac{1}{a\exp(x)+1} = \ln(1+1/a)$.]

b) Find the chemical potential in the limit $T = 0$.

8. (**Atomic Physics**) An atomic beam of initial velocity 100 m/s moves along the axis of a solenoid, starting in a region where the magnetic field is practically zero and moving into the center of the solenoid where the field is non-zero. The atoms have a mass of 108 atomic mass units (amu), a total electron spin *S=1/2*, a g-factor of 2, and orbital angular momentum *L=0*. The atomic beam is unpolarized. The atoms propagate along the *z*-axis, and the magnetic field of the solenoid is parallel to its axis and has the form $\vec{B} = B(z)\hat{z}$.

(a) List the relevant magnetic sublevels of the atoms' ground state. What are the force vectors acting on atoms in each of these states as a function of $B(z)$?

(b) Determine the velocities of atoms in each magnetic sublevel at a point inside the solenoid where the magnetic field $B(z)$ is 10 Tesla.

Note: 1 amu = $1.66\times10^{-27}\,\text{kg}$, and 1 Bohr magneton = $1\mu_B = \dfrac{e\hbar}{2m_e} = 9.27\times10^{-24}\,\text{A}\cdot\text{m}^2$.

Optional (do 2 of 4)

9. (**Nuclear Physics**): The half-life of Uranium-238 is about τ = 4.47 x 10^9 yr. Estimate the fraction f_U of the Earth (by mass) that would have to be Uranium-238 at the present time in order for the radioactive heating from its decays to equal the energy input from the Sun. (A single Uranium-238 decay releases 4.3 MeV, and the solar luminosity is 4 x 10^{26} J/sec, leading to an average surface temperature on the Earth of ~300 K. You may find other useful numbers at the front of the exam.)

10. (**Atomic Physics**) An atom with spin S=1/2 (g-factor = 2) resides in a magnetic field of 1 Tesla that is pointing in the z-direction.

(a) What are the magnetic energy levels of the atom in GHz?

(b) What type of electromagnetic transition couples these states? (Choose from E1, E2, M1, M2 etc.)

(c) Assume an initial spinor state $\left|\uparrow\right\rangle = \begin{pmatrix} c_u = 1 \\ c_d = 0 \end{pmatrix}$ at time *t=0*. We apply a weak oscillatory transverse radio-frequency magnetic field $\vec{B} = B_{RF}\cos(\omega t)\hat{x}$ that has a fixed, small amplitude B_{RF} and an angular frequency <u>that resonantly couples the spin states</u>. What is the Hamiltonian? (You may write it as a 2 x 2 matrix).

(d) Calculate the time-dependent coefficients $\begin{pmatrix} c_u(t) \\ c_d(t) \end{pmatrix}$.

Hint for d): It may be useful to set $\begin{pmatrix} c_u(t) \\ c_d(t) \end{pmatrix} = \begin{pmatrix} \tilde{c}_u(t)\exp(-i\mu_B Bt/\hbar) \\ \tilde{c}_d(t)\exp(+i\mu_B Bt/\hbar) \end{pmatrix}$ and to first solve for the tilde-coefficients. Neglect any rapidly oscillating terms in the differential equations you get (this is known as the rotating-wave approximation). Note also that 1 Bohr magneton $= 1\mu_B = \dfrac{e\hbar}{2m_e} = 9.27\times10^{-24}\,\mathrm{A{\cdot}m^2}$.

11. (**Particle Physics**) Answer the following:

a) The cross section to make a new particle at the LHC is 200 pb. This particle has three decay channels with branching ratios of 10%, 30%, and 60%. These decay channels have detection efficiencies of 2%, 0.3%, and 0.1%, respectively. How many of these new particles will be produced with an integrated luminosity of 100 fb^{-1}? About how many will be detected?

b) For a proton of rest mass 0.938 GeV/c^2 and with momentum 7 TeV/c, what is the speed of the proton? (These numbers are typical of the LHC.)

c) Which of the following processes are allowed and which are forbidden:

   i. $n \rightarrow p\, e^- \bar{\nu}_e$
   ii. $\mu^- \rightarrow e^- \bar{\nu}_e \nu_\mu$
   iii. $\mu^+ \mu^- \rightarrow \gamma\gamma$
   iv. $p\, e^- \rightarrow \gamma\gamma$
   v. $p\, p \rightarrow p\, p\, p\, \bar{p}$

12. (**Condensed Matter**) Consider a $2D$ material. The elastic deformation in this material can be described by a $2D$ vector field $\vec{u}(x, y)$. Here, x and y are the 2D coordinates and the x and y components of the vector field $\vec{u}(x, y)$ are functions of x and y: $u_x(x, y)$ and $u_y(x, y)$. Assume that the energy cost for a deformation $\vec{u}(x, y)$ in this $2D$ material is

$$E = \iint dx\, dy\, [C_1(\partial_x u_x - \partial_y u_y)^2 + C_2(\partial_x u_y + \partial_y u_x)^2] .$$

Here, C_1 and C_2 are positive constants.

(1) Distortions that cost no energy ($E = 0$) are known as zero-energy distortions. Prove that for any zero-energy distortion in this system, $u_x(x, y)$ and $u_y(x, y)$ must be harmonic functions. [Harmonic functions are functions that satisfy the Laplace equation $\nabla^2 f = 0$].

(2) In $2D$, harmonic functions can be written in terms of $f(x, y) = A\, e^{k_x x} \sin(k_y y + \varphi)$, where A, φ, k_x and k_y are constants. The physical meaning of this function $f(x, y)$ is that these zero-energy distortions are sine waves along the y direction but the amplitudes of these waves decay exponentially along the perpendicular direction, which is the x direction here. Find the relation between the wave vector k_y and the decay rate k_x.