University of Michigan Physics Department
Graduate Qualifying Examination

Part I: Modern Physics
Saturday, January 19, 2013   9 am – 2 pm

This is a closed book exam, but a number of useful quantities and formulas are provided in the front of the exam. (**Note that this list is more extensive than in past years.**) If you need to make an assumption or estimate, indicate it clearly. Show your work in an organized manner to receive partial credit for it. Answer the questions directly in this exam booklet. If you need more space than there is under the problem, continue on the back of the page or on additional blank pages that the proctor will provide. Please indicate if you continue your answer on another page.   If you use additional blank pages, state clearly at the top of each page your exam number, found at the upper right of this page—but not your name—the question number, and "page x of y" (if there is more than one page per question).

You must answer the first 8 required questions and 2 of the 4 optional questions. Indicate which of the latter you wish us to grade (e.g. by circling the question number). We will only grade the indicated optional questions. Good luck!!

Some integrals and series expansions

$$\int_{-\infty}^{\infty} \exp(-\alpha x^2)\, dx = \sqrt{\frac{\pi}{\alpha}}$$

$$\int_{-\infty}^{\infty} x^2 \exp(-\alpha x^2)\, dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\exp(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots$$

Some Fundamental Constants

speed of light $c = 2.998 \times 10^8$ m/s

proton charge $e = 1.602 \times 10^{-19}$ C

Planck's constant $\hbar = 6.626 \times 10^{-34}$ J·s $= 4.136 \times 10^{-15}$ eV·s

Rydberg constant $R_\infty = 1.097 \times 10^7 \, \text{m}^{-1}$

Coulomb constant $k = (4\pi\epsilon_0)^{-1} = 8.988 \times 10^9$ N·m^2 / C^2

vacuum permeability $\mu_0 = 4\pi \times 10^{-7}$ T·m/A

universal gas constant $R = 8.3$ J / K·mol

Avogadro's number $N_A = 6.02 \times 10^{23} \, \text{mol}^{-1}$

Boltzmann's constant $k_B = R/N_A = 1.38 \times 10^{-23}$ J/K $= 8.617 \times 10^{-5}$ eV/K

Stefan-Boltzmann constant $\sigma = 5.67 \times 10^{-8}$ W / m^{2}K^4

radius of the sun $R_{sun} = 6.96 \times 10^8 \, \text{m}$

radius of the earth $R_{earth} = 6.37 \times 10^6 \, \text{m}$

radius of the moon $R_{moon} = 1.74 \times 10^6 \, \text{m}$

gravitational constant $G = 6.67 \times 10^{-11} \text{m}^3 / (\text{kg·s}^2)$

Required (do all problems)

1. (**Quantum Mechanics**) An electron is at rest in an oscillating magnetic field

$$\vec{B} = B_0 \, \cos(\omega t) \, \hat{z}$$

where B_0 and ω are constants. At time $t = 0$, the electron spin is pointed along the $+x$ axis.

(a) Use the time-dependent Schrodinger equation to determine the spin state $\chi(t)$ at any subsequent time. (Here $\chi(t)$ is a 2-component spinor.) Make sure to normalize. (**Hint**: the Hamiltonian associated with the torque on the electron is $H = -\gamma \vec{B} \cdot \vec{S}$, where γ is the gyromagnetic ratio of the electron).

(b) If S_x is measured, what is the probability of measuring $-\hbar/2$ as a function of time?

(**Hint**: the eigen spinors of S_x are $\chi_+^x = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ and $\chi_+^x = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$, and the relation

$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$ will be useful.)

(c) What is the minimum field amplitude B_0 required to force a complete flip in S_x (i.e for there to be at least one instant of time t at which S_x has reversed direction with probability 1)?

2. (**Quantum Mechanics**) Show that if $[x,p] = i\hbar$, then $[f(x), p] = i\hbar(df / dx)$.

3. (**Quantum Mechanics**) A quantum particle of mass m is trapped in the ground state of an infinitely deep one-dimensional square well. The walls of the well are at $x = 0$ and at $x = L$. At time $t=0$ the right wall is suddenly shifted to $x = 2L$; you may consider the move to be infinitely fast.

(a) Determine the probabilities P_0 and P_1 of finding the particle in the lowest state and the first excited state, respectively, of the well after the wall shift.

(b) Neglect all probabilities in higher states (i.e. set $P_n = 0$ for $n > 1$ and normalize the probabilities P_0 and P_1 such that they add up to 1). What is the expectation value of the particle position relative to the middle of the (new) well at times $t > 0$? That is, what is $\langle x \rangle - L$ as a function of time for $t > 0$?

In part a), you may find this useful:

$$\text{For } a \neq b, \quad \int \sin(ax)\sin(bx)dx = \frac{\sin(x(a-b))}{2(a-b)} - \frac{\sin(x(a+b))}{2(a+b)}.$$

In part b), you may find this useful: $\displaystyle\int_{-L}^{L} \sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi x}{2L}\right) x \, dx = \frac{32L^2}{9\pi^2}.$

4. (**Quantum Mechanics**) Consider a simple harmonic oscillator with a small perturbation so that the Hamiltonian has the form

$$\mathcal{H} = \frac{1}{2}\left(\frac{p^2}{2m} + m\omega^2 x^2\right) + bx^4,$$

where all of the symbols have their usual meanings and where b is a small constant (so that the third term is much smaller than the first two). A photon is emitted when the system makes a transition from the *second* excited state down to the ground state.

   a) For the unperturbed system $b=0$, find the frequency of the photon.
   b) Using perturbation theory and working to leading order, find the frequency of the photon for $b \neq 0$.

5. (**Statistical Mechanics**) When Bose-Einstein condensates are created in the lab, magneto-optical traps are typically used to confine the condensing atoms. The single particle energy levels ("orbitals"), and thus the density of states, in these traps can be approximated by those for a harmonic oscillator.

In one dimension, the orbitals have energies $\varepsilon_n = n\hbar\omega$, with n = 0, 1, 2, 3,... and ω the oscillator's natural frequency. (We have subtracted off an irrelevant constant so that $\varepsilon_0 = 0$.)

   a) What is the density of states $D(\varepsilon)$ for such a one-dimensional harmonic oscillator? By definition, $D(\varepsilon)\mathrm{d}\varepsilon$ is the number of orbitals with energies between ε and $\varepsilon + \mathrm{d}\varepsilon$; note that, unlike some definitions you may have seen, the one used here does <u>not</u> divide by the system volume.

   b) Can non-interacting, spin 0 bosons in a one-dimensional harmonic potential undergo Bose-Einstein condensation? <u>Why or why not</u>? (Be sure to give a careful argument as to why or why not; answers that look like a random guess will not receive credit.)

The density of states for a 3-dimensional harmonic oscillator of frequency ω is

$$D(\varepsilon) = \frac{\varepsilon^2}{2(\hbar\omega)^3}$$

(where, as before, the ground state energy is 0).

   c) Give an expression for the Bose-Einstein condensation temperature T_E as a function of the number N of non-interacting, spin 0 bosons in a trap and the other parameters in the problem. You do not need to evaluate any integrals that are simply dimensionless numbers. (For the purposes of checking your answer, you may find it useful to know that the linear size of the trap goes like $1/\omega$, so you can think of $1/\omega^3$ as being propotional to the system volume.)

6. (**Statistical Mechanics**) Rubber consists of many cross-linked polymer chains, and as a consequence its elastic properties vary with temperature T in an unusual way. To understand the elastic properties of rubber, consider just one polymer chain. We model the chain as being composed of N rigid rods of length l connected to each other at their ends. All rods are identical. The angle between two adjacent rods is assumed to take one of only two possible values, 0 or π, independently at each junction. The two choices of the angle between successive rods have the same energy. The model is illustrated in the figure. One end of the chain is fixed at the origin $x = 0$, while the other is free to take any position along the x axis in discrete steps of length l.
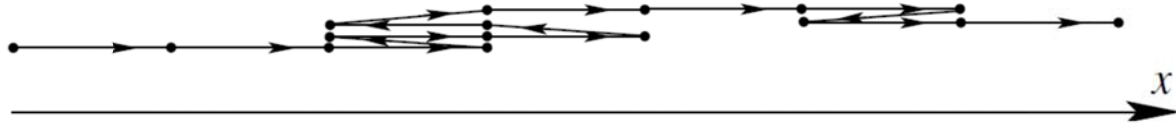

Figure: A simplified model of rubber. A rubber molecule is modeled by a chain consisting of N rigid identical rods of length l. The rigid rods are shown as arrows with circles at both ends. The rods are assumed to be always parallel to the x-axis, but may point in either of two directions.

a) When the chain's free end is at the position $x = L$, find an expression for the number of configurations Ω of the chain in terms of N, L, and l.

b) Find the entropy S of the chain under the same conditions and in terms of the same variables. Assume $N \gg 1$ and use Stirling's approximation for factorials of large numbers: $\ln(n!) \approx n(\ln n - 1)$.

c) Find the Helmholtz free energy F of the chain with the free end at $x = L$ as a function of N, L, l, and the temperature T.

d) Find the force f necessary to hold the free end at $x = L$ as a function of N, L, l, and T. **Hint**: The distance L is analogous to the volume of a gas, and the tension force f is analogous to the **negative** of the pressure in a gas (since an unconstrained polymer chain will tend to shrink, whereas a gas would tend to expand).

e) Find the asymptotic behavior of f for $L \ll Nl$. What is the effective spring constant for this chain in this limit?

7. (**Statistical Mechanics**) Consider a highly relativistic, but otherwise classical (i.e. non-quantum) ideal gas with N particles at temperature T in volume V. A particle with momentum $\mathbf{p}$ in such a gas has energy $c \, |\mathbf{p}| = c\sqrt{p_x^2 + p_y^2 + p_z^2}$. Express all quantities below in terms of N, T, V, c, and any other needed universal constants.

a) Find the gas's internal energy U.

b) Find the Helmholtz free energy F.

c) Find the pressure P.

d) Find the heat capacity at constant volume C_V .

8. (**Condensed Matter**) We know that the sound velocity $c = \sqrt{\frac{\partial P}{\partial \rho}}\Big|_S$. At zero temperature, the pressure of a free Fermi gas is $P = -\frac{\partial E}{\partial V}\Big|_N$ (in this partial derivative, the particle number N is fixed). Find the sound velocity in a free Fermi gas at $T = 0\ K$ in terms of the particle density.

Optional (do 2 of 4 problems)

9. (**Atomic Physics**) Level structure of Helium I

a) Sketch the energy level diagram for the lowest five energy levels of helium, ignoring fine structure splitting (i.e., show all possible levels for $n = 1$ and $n = 2$). Identify each level using spectroscopic notation. The levels must be drawn in correct relative order.

b) For each of the five levels, list the spectroscopic notation and the corresponding n, S and L quantum numbers in a well organized table. Add a column in which you show all possible J quantum numbers for each level.

c) Can you identify any metastable states? If so, which state(s) are these? Which one will have the longest lifetime?

10. (**Nuclear Physics**) Nuclide X decays into A and B with half-life T_X, and then A subsequently decays into C and D with half-life T_A. B, C and D are all stable. Assuming that at $t=0$ there are $N_X(0)$ X nuclides, but no other nuclides present, find the expressions for the number of A, B, C, D and X nuclides at some later time t.

11. (**Particle Physics**) Particle X is at rest and decays into particles A and B.

    a) Derive the velocity of particle A as a function of the masses of X, A, and B. Express this velocity also in the simplifying cases of…

    b) $M_A = 0$,

    c) $M_A = M_B = M_X / 2$, and

    d) $M_A = M_B = m$, where $m < M_X / 2$.

12. (**Condensed Matter**) Phospholipids are molecules with charged heads and hydrophobic tails. When they are mixed with water, they can form a variety of ordered phases. Two of the most common are the lamellar phase and the hexagonal phase. These resemble, respectively, a stack of equally spaced, planar lipid sheets and a hexagonally packed collection of long lipid rods (Fig. 1). Suppose that you perform small angle X-ray scattering (Fig. 2) on a polycrystalline sample of one of these lipid phases. (A polycrystalline sample is one that contains many small crystallites with different, effectively random, orientations. You may possibly have heard scattering from such a sample described as "powder diffraction.")
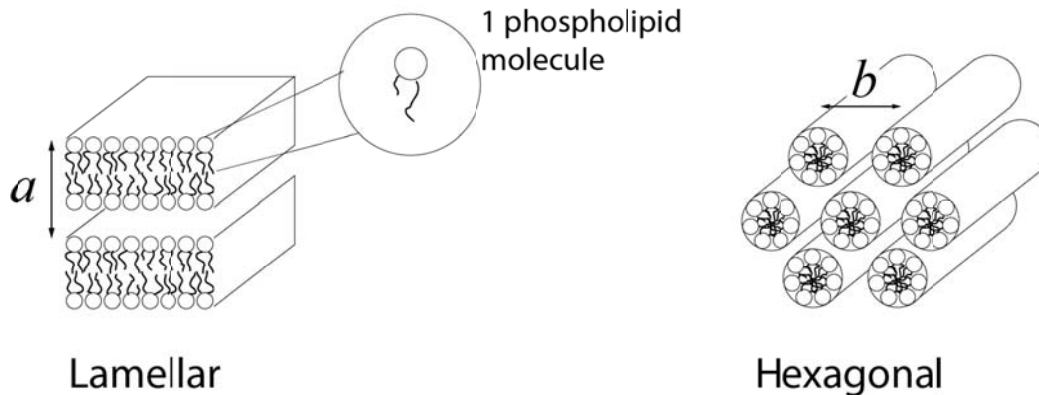

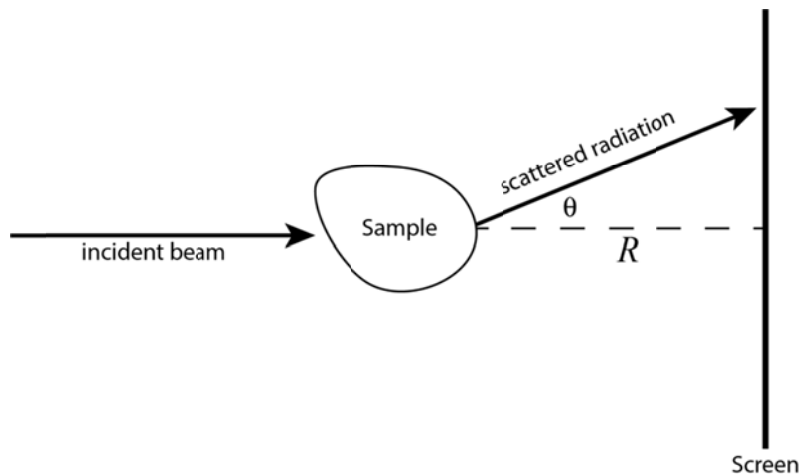## Figure 1: Lamellar and Hexagonal Phases

Figure 2: Small Angle X-Ray Scattering Setup

a) Describe the diffraction pattern you would see on a screen a distance R from the sample for a lamellar phase with lattice constant a. (You should describe the general pattern of diffraction peaks and state how the distances between these peaks are related to a and to the wavenumber q_{in} of the incident X-rays.)

b) Describe the diffraction pattern you would see for a hexagonal phase with lattice constant b.

c) How could you tell from the diffraction pattern whether you have a lamellar or a hexagonal phase *without* knowing their lattice constants ahead of time?