Name:_______________________________

Exam Number:_______

University of Michigan Physics Department
Graduate Qualifying Examination

Part II: Modern Physics

Saturday 17 Jan 2015 9:30 am – 2:30 pm

This is a closed book exam, but a number of useful quantities and formulas are provided in the front of the exam. If you need to make an assumption or estimate, indicate it clearly. Show your work in an organized manner to receive partial credit for it. Answer the questions directly in this exam booklet. If you need more space than there is under the problem, continue on the back of the page or on additional blank pages that the proctor will provide. Please clearly indicate if you continue your answer on another page. Label additional blank pages with your exam number, found at the upper right of this page (but not with your name). Also clearly state the problem number and "page x of y" (if there is more than one additional page for a given question).

You must answer the first 8 required questions and 2 of the 4 optional questions. Indicate which of the latter you wish us to grade (e.g. by circling the question number). We will only grade the indicated optional questions. Good luck!!

Some integrals and series expansions

$$\int_{-\infty}^{\infty} \exp(-\alpha x^2)\,\mathrm{d}x = \sqrt{\frac{\pi}{\alpha}}$$

$$\int_{-\infty}^{\infty} x^2 \exp(-\alpha x^2)\,\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\exp(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots$$

Some Fundamental Constants

speed of light $c = 2.998\times10^{8}$ m/s

proton charge $e = 1.602\times10^{-19}$ C

Planck's constant h $= 6.626\times10^{-34}$ J·s $= 4.136\times10^{-15}$ eV·s

Rydberg constant $R_{\infty} = 1.097\times10^{7}\,\mathrm{m}^{-1}$

Coulomb constant $k = (4\pi\varepsilon_0)^{-1} = 8.988\times10^{9}$ $\mathrm{N{\cdot}m^2 / C^2}$

vacuum permeability $\mu_0 = 4\pi\times10^{-7}$ T·m/A

universal gas constant $R = 8.3$ J / K·mol

Avogadro′s number $\mathrm{N_A} = 6.02\times10^{23}\,\mathrm{mol^{-1}}$

Boltzmann′s constant $\mathrm{k_B} = \mathrm{R/N_A} = 1.38\times10^{-23}$ J/K $= 8.617\times10^{-5}$ eV/K

Stefan-Boltzmann constant $\sigma = 5.67\times10^{-8}\,\mathrm{W / m^2K^4}$

radius of the sun $R_{\mathrm{sun}} = 6.96\times10^{8}$ m

radius of the earth $R_{\mathrm{earth}} = 6.37\times10^{6}$ m

radius of the moon $R_{\mathrm{moon}} = 1.74\times10^{6}$ m

gravitational constant $G = 6.67\times10^{-11}\,\mathrm{m^3 / (kg{\cdot}s^2)}$

1. Quantum Mechanics:

Consider a permanent planar quantum mechanical dipole rotor $\mathbf{p}$ that is confined to the x-y plane. For a free rotor, the Hamiltonian is given by

$$H_0 = -\frac{\hbar^2}{2I}\frac{\partial^2}{\partial\phi^2}\,,$$

where I is the moment of inertia and where ϕ is the angle measured with respect to the x-axis. We will also consider a small perturbation H_1 to the Hamiltonian due to an electric field $\mathbf{D}$ along the y-axis, where

$$H_1 = -\mathbf{p}\cdot\mathbf{D}\,.$$

a) Find all of the energy eigenvalues and eigenstates of H_0. For the sake of definiteness, arrange for the states to be eigenstates of the angular momentum operator $L_z = -i\hbar(\partial/\partial\phi)$.

b) Find all of the matrix elements (diagonal and off-diagonal) of the perturbed energy operator H_1 between the original eigenstates (of H_0) found in part (a).

c) Find the first order corrections to the energy levels.

2. Quantum Mechanics:
Consider a free particle having mass m with initial wave function $\psi(x,0)$ of the form

$$\psi(x,0) = \frac{1}{2\sqrt{\sigma}} \begin{cases} 1 & |x+a| \le \sigma \\ 1 & |x-a| \le \sigma \\ 0 & \text{otherwise} \end{cases} .$$

where $\sigma \ll a$ and both σ and a are real and positive.

a) Sketch $|\psi(x,0)|^2$ and prove that $\psi(x,0)$ is normalized correctly.

b) Using the uncertainty principle, estimate the earliest time at which interference between the two parts of the wave packet become important.

c) Calculate the $k-$space amplitude $a(k)$ and calculate the $k-$space distribution function $|a(k)|^2$. Show that the distribution function in k space resembles that of the spatial distribution function for two slit diffraction. [Note: $\int_a^b e^{-ikx} dx = \left(\frac{e^{-ikb} - e^{-ika}}{-ik}\right)$]

d) Write an integral expression for $\psi(x,t)$.

3. Statistical Mechanics:

In this problem you will examine the properties of blackbody radiation in one dimension. Consider photons in a one dimensional cavity of length L. The Hamiltonian is $H = \sum_i c|p_i|$, where p_i is the momentum of the i^{th} photon.

a) Calculate the density of states $g(\epsilon)$, where ϵ is the energy of the state.

b) Calculate the average internal energy and specific heat C_V as functions of L and the temperature T. You may need the following integral $\int_0^\infty \frac{x\,dx}{(e^x-1)} = \frac{\pi^2}{6}$.

c) Calculate the entropy S, Helmholtz free energy, and pressure p.

4. Particle Physics:

A top quark with mass $m_t = 172$ GeV/c^2 decays to a b quark with $m_b = 5$ GeV/c^2 and a W boson with $m_W = 80$ GeV/c^2.

a) What force mediates this interaction?

b) In the rest frame of the decaying top quark, what is the energy of the W boson in GeV?

5. Quantum Mechanics:

a) Assume $|\psi\rangle$ simultaneously obeys the following equations:

$$\begin{aligned} J_x |\psi\rangle &= a_x |\psi\rangle \\ J_y |\psi\rangle &= a_y |\psi\rangle \\ J_z |\psi\rangle &= a_z |\psi\rangle , \end{aligned}$$

where J_i represent the components of the angular momentum operator. Find the values of a_x, a_y, a_z. (Hint: use the commutation relation).

b) If we add four spin-1/2 operators, i.e. $\vec{S} = \vec{S}_1 + \vec{S}_2 + \vec{S}_3 + \vec{S}_4$, what are the possible eigenvalues of the operator $\vec{S}^2$? How many orthogonal states $|\Psi\rangle$ of the four spins have the property $\vec{S}^2 |\Psi\rangle = 0$.

6. Statistical Mechanics:

Consider a system of spin-1/2 particles with magnetic moment μ in a uniform magnetic field of strength B. The system thus has two energy states $\pm E$ corresponding to spins that are aligned and anti-aligned with the background field. Assume that the system is in contact with a thermal reservoir of temperature T. Electromagnetic radiation impinging on the system can be absorbed if the frequency of the radiation satisfies the Bohr condition.

a) Write down the constraint on the frequency required for the incoming radiation to be absorbed.

b) The power absorbed from the radiation field P_{ab} is proportional to the difference between the number of particles in the two energy states. Find the temperature dependence of the absorbed power P_{ab}. Evaluate the result in the high temperature limit $kT \gg E$.

7. Statistical Mechanics:

Consider a volume of size $2V$ split in two equal parts. On one side of the partition, there are N molecules of oxygen gas held at temperature T. On the other, N molecules of nitrogen are held at the same temperature. The energy of a molecule can be written as:

$$E_k^i = \frac{\vec{p}^2}{2m} + \epsilon_k^i$$

where ϵ_k denotes the energy levels corresponding to the internal states of the gas, and the i index reminds us that the internal energy levels may differ for the different types of molecule.

a) Evaluate the free energy F for the oxygen gas as it resides separately on its side of the partition. You should display explicitly any dependence on V, but if you encounter a function solely of the temperature you should parametrize it as $f(T)$. The following integral may be useful:

$$\int_{-\infty}^{\infty} e^{-y^2/a} = \sqrt{\pi a}$$

b) Calculate the entropy of the oxygen gas. You answer should be in terms of N, V, and (the derivative of) your unknown function $f(T)$.

c) The gases are now allowed to mix, but no heat is exchanged with the surroundings. Neglect interactions between the molecules. Does the temperature change? If so, by how much? Does the total entropy of the system change? If so, by how much?

8. Quantum Mechanics:

Use the WKB approximation to determine how the energy levels of a potential $V(x) = a\,|x|^{\mu}$ $(\mu > 0)$ vary with quantum number n for large n. Note, it is not necessary to explicitly evaluate any integral in this problem. What are the only values of μ for which the the energy does not depend on fractional powers of n? Physically to what potentials do these values of μ correspond?

9. Atomic Physics (Optional) :
For the ground state of the hydrogen atom, the only nonvanishing contribution to the hyperfine structure comes from the component of the magnetic field of the proton given by $\mathbf{B}_{contact} = \frac{2\mu_0}{3}\mu_p\delta(\mathbf{r})$, where μ_p is the magnetic moment of the proton,

$$\mu_p = \frac{eg_p\mathbf{S}_p}{2m_p} \tag{1}$$

and $\mathbf{S}_p$ is the spin of the proton. This field gives rise to an interaction energy with the electron given by

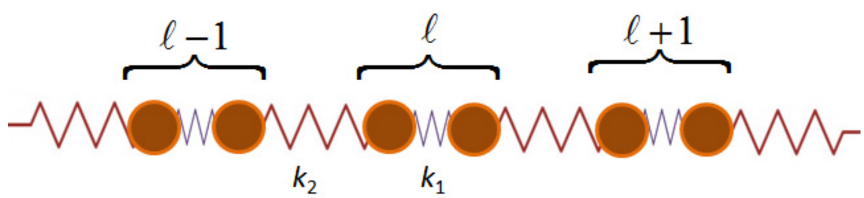
$$H'_{hf} = -\mu_e \cdot \mathbf{B}_{contact}. \tag{2}$$

Using the fact that the g factor of the proton is 5.59 and that the proton has spin 1/2, calculate the hyperfine splitting in the ground state of hydrogen in frequency and wavelength units. You can use the fact that $\psi_{100}(r) = (1/\pi a_0^3)^{1/2}e^{-r/a_0}$ and $\delta(\mathbf{r}) = \delta(x)\delta(y)\delta(z)$.

10. Condensed Matter (Optional):
Consider a dimerized linear chain where all the atoms are identical, but $k_1 \neq k_2$. M is the mass of the atoms and k_1, k_2 are the spring constants connecting two neighboring atoms.

a) Find and plot the phonon dispersion (ω vs. q) for longitudinal modes propagating along the chain.

b) repeat for $k_1 = k_2$.



11. Condensed Matter (Optional):
Consider a powder, that is, a random distribution of crystallites of a material that cystallizes in the face-centered cubic (fcc) structure with lattice spacing $a = 3.55$ Å. Calculate the positions of the first three x-ray Bragg diffraction peaks in terms of the quantity $K = 4\pi \sin\theta/\lambda$, where λ is the wavelength of the x-ray radiation and 2θ is the scattering angle (the angle between the incident and scattered x-ray beams).

12. Nuclear Physics (Optional):
A neutrino detector was composed of a large (615 metric tons = 6.15 $\times 10^5$ kg) tank of tetrachloroethylene (C_2 Cl_4) located in a mine. Solar neutrinos incident on Chlorine atoms converted these atoms to Argon atoms.

a) Write down an equation summarizing the reaction. What else is produced besides an Argon nucleus?

b) The mass of a neutral Cl^{37} atom is 34433.52 MeV/c^2, and the mass of a neutral Ar^{37} atom is 34434.33 MeV/c^2. What is the threshold neutrino energy for this process to proceed?

c) Notably, there are neutrinos produced in sun via the beta decay of B^8 with energies all the way up to 15 MeV with a flux at the Earth of $\Phi_\nu^{B^8} = 1.4 \times 10^7\,\mathrm{cm}^{-2}\,\mathrm{sec}^{-1}$. (these neutrinos are part of the pp-chain) The cross section for these B^8 neutrinos with the Cl atoms is $\sigma_{\nu Cl} = 1.14 \times 10^{-42}$ cm^2. The atomic mass of carbon is 12.01, and the atomic mass of Chlorine is 35.45. Chlorine-37 makes up 24% of naturally occurring Chlorine. What is the average number of Argon atoms that can be expected to be produced by these B^8 neutrinos in a year?

d) These Argon atoms are then chemically separated from the detector, and counted using the fact that the created Argon isotope is unstable with half-life = 35 days. Given a sample of 10 radioactive Argon atoms, how long should the sample be observed such that you would have at least a 99% chance that the Ar atoms would have all decayed?