Extension of Post's Lattice to Countable-Borel Clones

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July 9, 2023

Abstract

Given a cardinal κ and an underlying set X, a κ-ary *function clone* on X is a set of functions $f : X^n \to X$ for $n < \kappa$, which contains all projection functions and is closed under composition. In 1941, Emil Post fully characterized all clones of finite functions $f : 2^n \to 2$ in a lattice famously titled *Post's lattice*, ordered by inclusion. We seek to extend the view of clones to include countably-infinite Borel functions and characterize this extended notion of Post's lattice in terms of the preservation of countable relations.

Acknowledgement

I'd like to thank my mentor, Ronnie Chen, for not only providing me with this great research opportunity, but also for his guidance throughout the project and his patience throughout the times I got lost (and through my very long stories). This experience wouldn't have been nearly as fantastic without him.

1 Introduction

In propositional logic, all propositional formulas, up to logical equivalence, can be considered as functions $f : 2^n \to 2$ where $2 = \{0, 1\}$. In Post's lattice, Emil Post characterized all 'classes' of propositional logic, where various restrictions are in place for propositional formulas, such as positive logic where there are no negations. As shown in fig. 1 below, there are countably many such classes. Before discussing Post's lattice and classes, we introduce some preliminary functions and notation.

1.1 Preliminaries

For two elements $a, b \in 2$, define

$$a \wedge b \coloneqq \min(a, b), \qquad a \vee b \coloneqq \max(a, b), \qquad \neg a \coloneqq 1 - a.$$

Two other functions of note are $0(\vec{x})$ and $1(\vec{x})$, defined over any 2^n, which maps any n-tuples to 0 or 1, respectively. Typically, when referring to functions, inputs are omitted (i.e. '$\wedge$' instead of '$a \wedge b$').

In 2^n, $\vec{a}^n$ refers to an n-tuple of elements in 2, and f^n refers to a function $f : 2^n \to 2$. Lastly, a partial order $\leq$ is defined on 2^n by $\vec{a} \leq \vec{b} \iff a_i \leq b_i \ \forall i < n$. This ordering is not a total order.

1.2 Post's Lattice

In fig. 1 below, each point in the lattice is a class of functions, ordered upward by inclusion. The top class, $\mathbb{P}_2^{<\omega}$ is the class of all functions, and $\langle \varnothing \rangle$ is the smallest class, the class of atomic formulas modulo logical equivalence. There are five classes directly below $\mathbb{P}_2^{<\omega}$. M, the class of monotone functions. Modulo logical equivalence, this consists of all positive propositional formulas. T_0 and T_1 are the classes where $f(\vec{0}) = 0$ and $f(\vec{1}) = 1$, respectively. D is the class where $f(x) = \neg f(\neg \vec{x})$. A is the class of linear functions. These are referred to as the 'maximal' classes of Post's lattice.

We seek to extend the notion of Post's lattice to capture countably-infinite propositional logic. In order to do so, we must properly define 'classes,' called *clones*.

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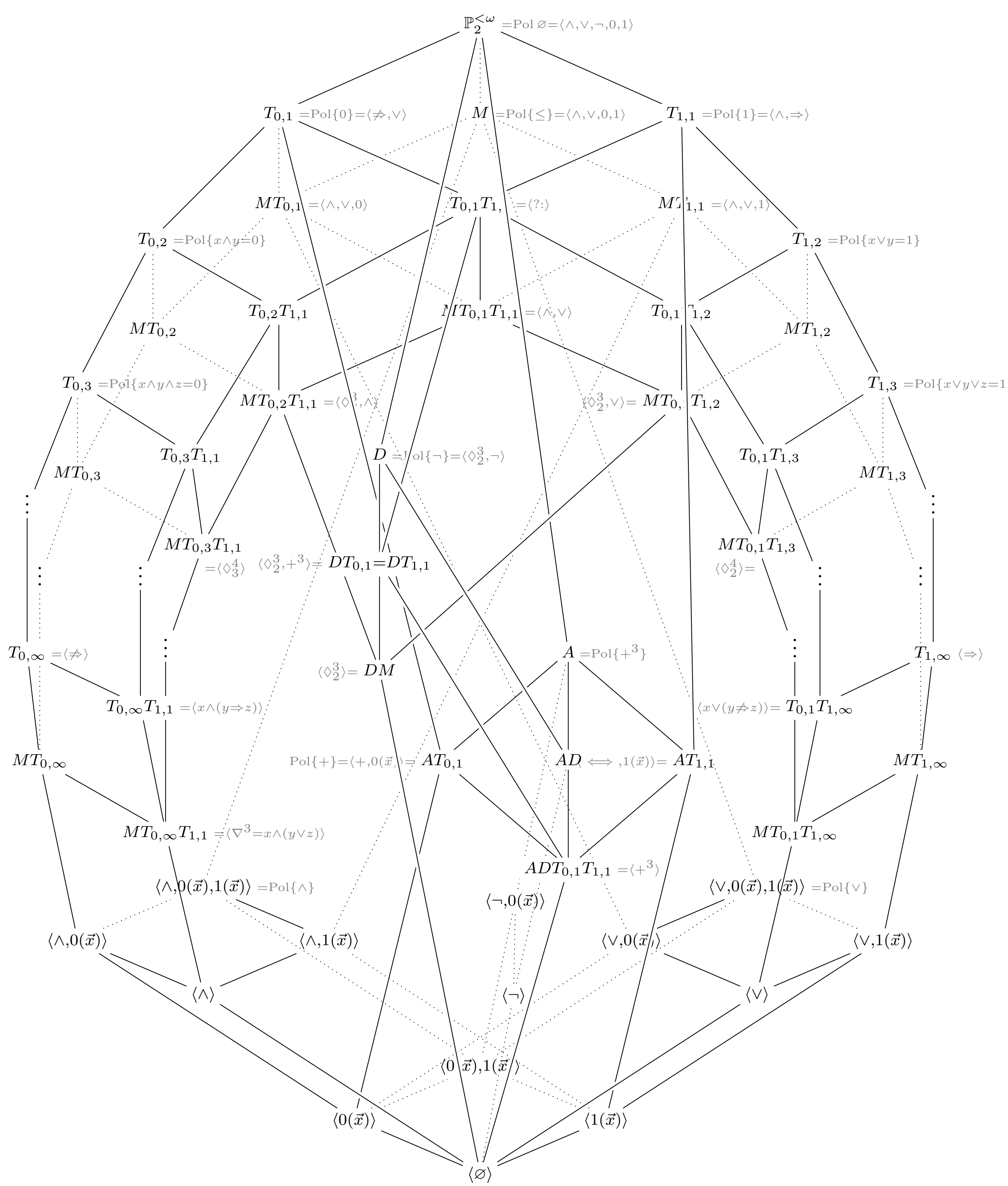
Figure 1: Post's Lattice



The sides of the lattice are symmetrical. This is due to the unique nontrivial isomorphism between clones, $\delta(f(\vec{x})) = \neg f(\neg\vec{x})$ [2, Theorem 9.2.15]. For any function f, $\delta(f)$ is referred to as the 'De Morgan dual' of f. For example, $\delta(\wedge) = \vee$.

2 Clones

There are two dominating perspectives towards clones. One of them is in terms of clones being generated by sets of functions, and another is in terms of clones preserving relations. This section will introduce clones and develop these two views.

2.1 Background

Let κ be a cardinal and X be a set. Denote

$$\mathbb{P}_X^{<\kappa} := \bigsqcup_{n<\kappa} X^{X^n},$$

as the set of all functions $f : X^n \to X$ where $n < \kappa$.

Definition 2.1 (Clones)**.** A $<\kappa$*-ary clone* on X is a subset $F \subseteq \mathbb{P}_X^{<\kappa}$ that contains projection functions

$$\begin{aligned} F \ni \pi_i^n : X^n &\to X \\ \vec{x} &\mapsto x_i \end{aligned}$$

where $i \leq n < \kappa$, and is closed under composition. In other words, for $f^n, 0 \leq i < n$, and $g_i^m \in F$,

$$\begin{aligned} F \ni f \circ \vec{g} : X^m &\to X \\ \vec{x} &\mapsto f((g_i(\vec{x}))_{i<n}). \end{aligned}$$

In other words, a clone is a substructure of the multi-sorted (infinitary) algebraic structure $\mathbb{P}_X^{<\kappa}$ with constants π_i^n and operations $\circ$. A function f being '$<\kappa$-ary' means $f : X^n \to X$ for some $n < \kappa$.

Example (Post's Lattice Clones; $X = 2, \kappa = \omega$)**.**

- $\mathbb{P}_2^{<\omega}$ is the clone of all functions in Post's lattice. Modulo logical equivalence, this consists of all propositional formulas, with each π_i^n acting as an atomic formula.
- $M^{<\omega}$ is the clone of finite, all monotone functions; $\vec{a}^n \leq \vec{b}^n$ implies that for f^n, $f(\vec{a}) \leq f(\vec{b})$.
- $T_0^{<\omega}$ is the clone of finite, 0-preserving functions; $f^n(\vec{0}^n) = 0$.

Let κ be a cardinal. Denote the set of $<\kappa$-ary clones on X as

$$\mathrm{Clo}(X) = \mathrm{Clo}^{<\kappa}(X) \subseteq \mathcal{P}(\mathbb{P}_X) = \mathcal{P}\left(\bigsqcup_{n<\kappa} X^{X^n}\right) \cong \prod_{n<\kappa} \mathcal{P}\left(X^{X^n}\right).$$

$\mathrm{Clo}^{<\kappa}(X)$ is a lattice, ordered by set inclusion, and is closed under arbitrary intersection. Thus, any subset $A \subseteq \mathbb{P}_X$ has a smallest clone containing it, denoted $\langle A\rangle^{<\kappa}$, the *clone generated by* A. A is called a *generating set* of $\langle A\rangle^{<\kappa}$. When listing generated clones, braces are typically omitted; $\langle f\rangle$ is written instead of $\langle\{f\}\rangle$.

Lemma 2.1. *If $F \subseteq \mathbb{P}_X^{<\kappa_1}$ is a $<\kappa_1$-ary clone and $\kappa_2 \geq \kappa_1$, the $<\kappa_2$-ary clone $\langle F\rangle^{<\kappa_2}$ restricts back to F,*

$$\langle F\rangle^{<\kappa_2} \cap \mathbb{P}_X^{<\kappa_1} = F.$$

In fact, $\langle -\rangle^{<\kappa_2}$ has $(-) \cap \mathbb{P}_X^{<\kappa_1}$ as a retraction.

$$\mathrm{Clo}^{<\kappa_1}(X) \underset{(-)\cap\mathbb{P}_X^{<\kappa_1}}{\overset{\langle -\rangle^{<\kappa_2}}{\rightleftarrows}} \mathrm{Clo}^{<\kappa_2}(X) \tag{1}$$

By definition, $\langle -\rangle^{<\kappa_2}$ is left-adjoint to the restriction map,

$$\langle F\rangle^{<\kappa_2} \subseteq G \iff F \subseteq G \cap \mathbb{P}_X^{<\kappa_1}.$$

The images of clones under $\langle -\rangle^{<\kappa_2} : \mathrm{Clo}^{<\kappa_1}(X) \to \mathrm{Clo}^{<\kappa_2}(X)$ are called *essentially* $<\kappa_1$*-ary* clones. Note that $\langle K\rangle^{<\kappa_2}$ is the smallest $<\kappa_2$-ary clone that restricts to K.

Example (Generating Sets of Clones)**.**

- $\mathbb{P}_2^{<\omega} = \langle \wedge, \vee, \neg \rangle^{<\omega} = \langle \wedge, \neg \rangle^{<\omega}$,
- $\langle \mathbb{P}_2^{<\omega} \rangle^{<\omega_1} = \langle \wedge, \neg \rangle^{<\omega_1}$,
- $M^{<\omega} = \langle \wedge, \vee, 0(x), 1(\vec{x}) \rangle^{<\omega}$,
- $T_0^{<\omega} = \langle \not\Rightarrow, \vee \rangle^{<\omega}$.

Each clone is clearly generated by some generating set A. Generating sets can help describe how clones are formed. In propositional logic, it is known that all statements can be built from $\wedge, \vee$ and $\neg$, as described by the above generating set for $\mathbb{P}_2^{<\omega}$.

As said before, $M^{<\omega}$ and $T_0^{<\omega}$ are 'maximal' in Post's lattice. A clone $C \subseteq K$ is *maximal* with respect to K if for any function $f \in K \setminus C$, $\langle C \cup \{f\} \rangle = K$.

The other main approach towards clones is through the preservation of relations, such as $\leq$ in the case of monotone functions. Functions which preserve relations are called *polymorphisms* of said relations.

2.2 Polymorphisms

Suppose $f : X^n \to X$ is an n-ary function, and $R \subseteq X^p$ is a p-ary relation for cardinals n and p. The following are synonymous:

- f *preserves* R,
- f is a *polymorphism* of R,
- f is a homomorphism from the product structure $(X, R)^n \to (X, R)$,
- R is *closed* under f,
- R is a substructure of the product structure $(X, f)^p$.

More precisely, for any $n \times p$ matrix $(x_t^i)_{i<n, t<p}$ of elements in X, if $\vec{x}^i = (x_t^i)_{t<p} \in R$ for each i, then $f((\vec{x}^i)_{i<n}) := (f((x_t^i)_{i<n}))_{t<p} \in R$.

Example (Polymorphism of a Relation)**.** Let $f^3(\vec{x}) = (x_0 \wedge x_1) \vee x_2$. This is a monotone function, or a polymorphism of $\leq$. Take $(0,1,0) \leq (0,1,1)$, This is expressed as

$$\begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

where each row is an element of $\leq$, or in other words, the first element of the row is less than or equal to the second. Map each column under f; $f(0,1,0) = 0, f(0,1,1) = 1$. $\begin{bmatrix} 0 & 1 \end{bmatrix} \in \leq$ as $0 \leq 1$.

Definition 2.2 (Pol and Inv)**.** Let P be a set of cardinals. For a set of P-ary relations $\mathcal{R}$, $\mathrm{Pol}^{<\kappa}\mathcal{R}$ is the set of $<\kappa$-ary functions which are polymorphisms of all relations in $\mathcal{R}$. Similarly, given a set of $<\kappa$-ary functions $\mathcal{F}$, $\mathrm{Inv}^P\mathcal{F}$ is the set of P-ary relations which are preserved by all functions in $\mathcal{F}$–the P-ary *inversions* of $\mathcal{F}$.

If $\mathcal{R} \subseteq \mathcal{T}$ are sets of P-ary relations, then $\mathrm{Pol}^{<\kappa}\mathcal{T} \subseteq \mathrm{Pol}^{<\kappa}\mathcal{R}$. This is analogously true for Inv^P. In fact, $\mathrm{Pol}^{<\kappa}$ and Inv^P form a Galois connection.

Significant to the study of clones is that for any set of P-ary relations $\mathcal{R}$, $\mathrm{Pol}^{<\kappa}\mathcal{R}$ is a $<\kappa$-ary clone. Thus, an alternative view of clones is as polymorphisms of relations.

Example (Clones as Polymorphisms)**.**

- $\mathbb{P}_2^{<\omega} = \mathrm{Pol}^{<\omega}\varnothing$,
- $M^{<\omega} = \mathrm{Pol}^{<\omega}\{\leq\}$, hence why $M^{<\omega}$ are called the monotone functions,
- $T_0^{<\omega} = \mathrm{Pol}^{<\omega}\{\vec{0}\}$, hence the name '$\vec{0}$-preserving functions.'

For two cardinals $\kappa_1 \leq \kappa_2$, dual to the fact that for a $<\kappa_1$-ary clone K, $\langle K \rangle^{<\kappa_2}$ is the smallest $<\kappa_2$-ary clone that restricts to K, if $K = \mathrm{Pol}^{<\kappa_1}\mathcal{R}$, then $\mathrm{Pol}^{<\kappa_2}\mathcal{R}$ is the largest $<\kappa_2$-ary clone that restricts to K.

3 The Countable-Borel Extension of Post's Lattice

In Post's Lattice, there are only finite functions $f : 2^n \to 2$. However, when extending to include functions $f : 2^\omega \to 2$ in countably-infinite propositional logic, not all infinite functions can be expressed; only Borel functions can be described.

3.1 Borel Functions in Cantor Space

In 2 with the discrete topology, every set is closed and open (clopen), and so any continuous function $f : 2^\omega \to 2$ has a clopen preimage in Cantor space. In 2^n, the finite product of 2, every set is necessarily clopen and thus every function is continuous. Due to the topology on Cantor space, the continuous functions $f : 2^\omega \to 2$ are exactly the functions that depend on finitely many variables, called 'essentially-finite,' which are re-expressed as finite functions $f : 2^n \to 2$. Thus, the 'properly-infinite' functions are the discontinuous ones. In countably-infinite propositional logic, every formula is expressed by countably-infinite 'ands', finite 'ands', countable 'ors', finite 'ors', and negations. Infinite 'and' and infinite 'or' are expressed as

$$\bigwedge^{\omega}(\vec{x}) \coloneqq x_0 \wedge x_1 \wedge \dots \qquad \bigvee^{\omega}(\vec{x}) \coloneqq x_0 \vee x_1 \vee \dots .$$

Cantor space also has natural extensions of meets, joins, and the order in 2^n by the following,

$$\vec{a}^{\omega} \wedge \vec{b}^{\omega} \coloneqq (a_0 \wedge b_0, a_1 \wedge b_1, \dots) \qquad \vec{c}^{\omega} \leq \vec{d}^{\omega} :\iff a_i \leq b_i \ \forall i \in \mathbb{N}.$$

The order defined here is also not a total order.

In Cantor space, the basis of the topology consists of the clopen sets within it, and so the Borel sets form a collection of sets inductively obtained from countable unions and intersections of clopen sets and other Borel sets defined before them. With respect to preimages of Borel functions $f : 2^\omega \to 2$, $\bigwedge$, $\bigvee$, and $\neg$ correspond to countable unions, intersections, and complements.

Example (Borel Functions)**.** Let

$$h^{\omega}(\vec{x}) = \bigwedge_{i \in \mathbb{N}} g_i^{\omega}(\vec{x})$$

Where $g_i(\vec{x})$ are Borel. $h(\vec{x}) = 0$ if and only if at least one of $g_i(\vec{x}) = 0$. Thus,

$$h^{-1}(0) = \bigcup_{i \in \mathbb{N}} g_i^{-1}(0)$$

Ultimately, the set of Borel $f : 2^\omega \to 2$ and the finite functions identified with continuous functions, denoted $\mathbb{B}$, consist of all finite and countably-infinite propositional formulas. Thus,

$$\mathbb{B} = \left\langle \bigwedge, \bigvee, \neg \right\rangle^{<\omega_1} = \left\langle \bigwedge, \neg \right\rangle^{<\omega_1} .$$

The focus of our study will be the lattice of subclones of $\mathbb{B}$, the countable-Borel lattice.

Since $\langle \mathbb{P}_2^{<\omega} \rangle^{<\omega_1} \subseteq \mathbb{B} \subseteq \mathbb{P}_2^{<\omega_1}$, every subclone of $\mathbb{B}$ restricts to a clone in Post's lattice. Thus, the extended lattice can be viewed in terms of sublattices of clones (ordered by inclusion) that restrict to a clone $\mathrm{Pol}^{<\omega}\mathcal{R} = K \subseteq \mathbb{P}_2^{<\omega}$, as illustrated by fig. 2. The rest of this text will focus on developing and describing fig. 2.

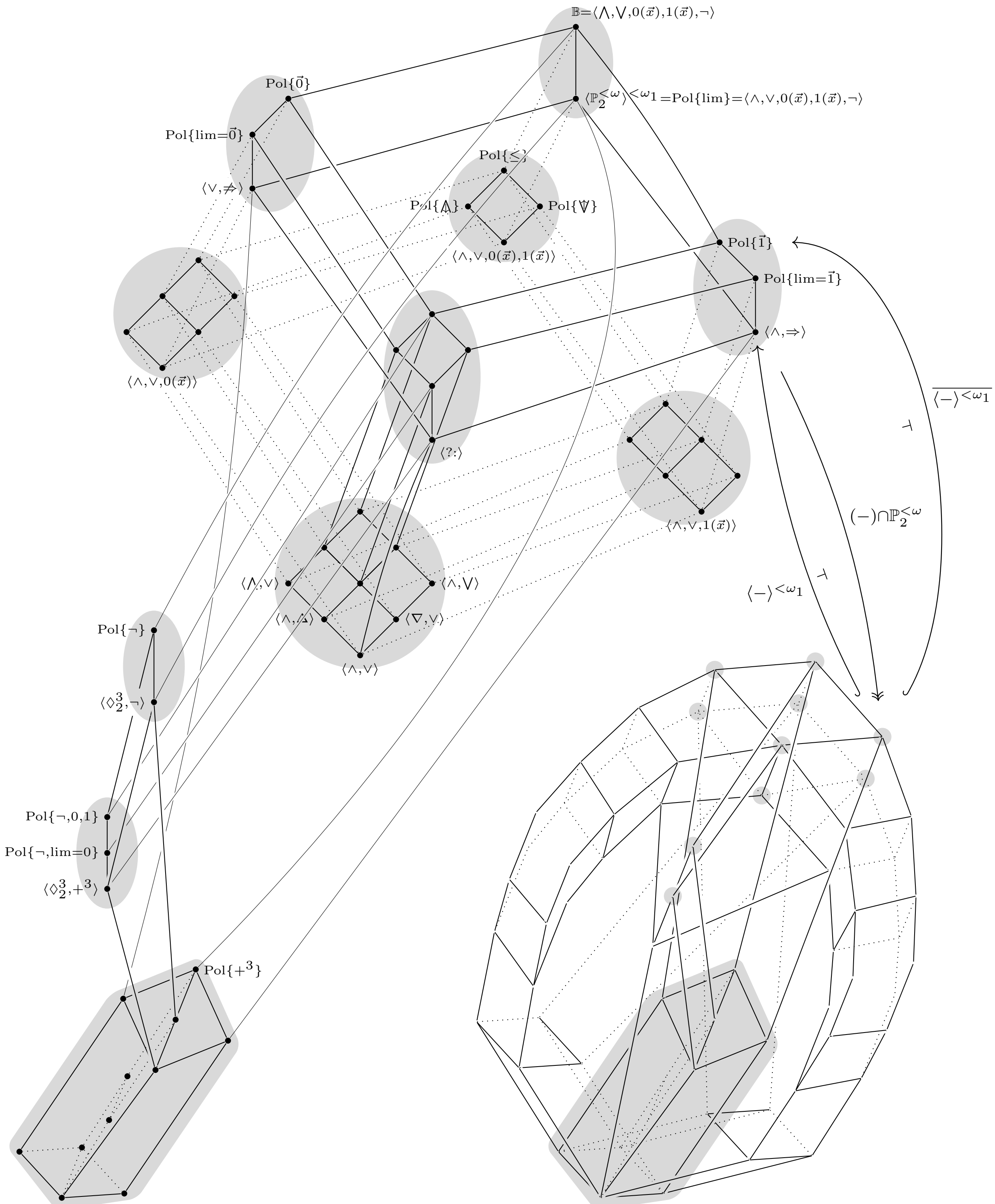
Given a set of countable-Borel clones that restrict to a Post clone K, immediately, $\langle K \rangle^{<\omega_1}$ and $\mathrm{Pol}^{<\omega_1}\mathcal{R}$ restrict to K. The former–called the *countable-Borel clone generated by K*–is the minimal element of the sublattice, and the latter is the maximal. Such maximal elements are denoted with a bold font, such as $\mathbb{K}$ in this instance.

The following lemma, stated without proof, is crucial to many forthcoming arguments.

Lemma 3.1. *If $F \subseteq \mathbb{B}$ is a clone, then each $f^\omega \in F$ is a limit in 2^{2^ω} of essentially-finitary operations in F.*

Since all clones in the countable-Borel lattice are 'extensions' of clones within Post's lattice, we can focus our view onto specific sections of the lattice.

Figure 2: The Countable-Borel Extension of Post's Lattice



$\overline{\langle - \rangle^{<\omega_1}}$ is the topological closure of $\langle - \rangle^{<\omega_1}$.

3.2 Maximal Clones

Again, the maximal clones of Post's lattice are

- $T_0^{<\omega} = \mathrm{Pol}^{<\omega}\{\vec{0}\}$,
- $T_1^{<\omega} = \mathrm{Pol}^{<\omega}\{\vec{1}\}$,
- $M^{<\omega} = \mathrm{Pol}^{<\omega}\{\leq\}$,
- $D = \mathrm{Pol}^{<\omega}\{\neg\}$,
- $A = \mathrm{Pol}^{<\omega}\{+^3\}$.

From Lemma 3.1, with the exception of the last of the following, the maximal clones of $\mathbb{B}$ are $\mathbb{T}_0 = \mathrm{Pol}^{<\omega_1}\{\vec{0}\}$, $\mathbb{T}_1 = \mathrm{Pol}^{<\omega_1}\{\vec{1}\}$, $\mathbb{M} = \mathrm{Pol}^{<\omega_1}\{\leq\}, \mathbb{D} = \mathrm{Pol}^{<\omega_1}\{\neg\}$, and $\langle \mathbb{P}_2^{<\omega}\rangle^{<\omega_1}$. Of particular interest are the clone of finite functions and the absence of any extension of the affine functions. The essence of the following argument can be applied to many of the following clones in the text,

Theorem 3.1 (Base Case of Wadge's Lemma)**.** *The clone of essentially-finite functions in $\mathbb{B}$, $\langle \mathbb{P}_2^{<\omega}\rangle^{<\omega_1}$, is a maximal clone.*

Proof. Fix $f \in \mathbb{B} \setminus \langle \mathbb{P}_2^{<\omega}\rangle^{<\omega_1}$, and let $G = \langle \{f\} \cup \mathbb{P}_2^{<\omega}\rangle^{<\omega_1}$. Since f is properly-infinite, it is discontinuous, implying that $f^{-1}(0)$ or $f^{-1}(1)$ is not closed. Since $\neg \in G$, without loss of generality, $f^{-1}(0)$ is not closed. There exists a sequence $(\vec{x}_n) \subseteq f^{-1}(0)$ that converges to a boundary point $\vec{x}_\infty \in f^{-1}(1)$. Then, there exists a continuous function $g : 2^\omega \to 2^\omega$ such that $(\underbrace{1, \ldots, 1}_{n}, 0, 0, \ldots) \mapsto \vec{x}_n$, and $\vec{1} \mapsto \vec{x}_\infty$. Then, each $\pi_i \circ g \in G$, and thus $\bigwedge = f \circ g \in G$, which implies that $G = \mathbb{B}$. □

Affine Functions

Not only is there no natural 'extension' of the affine clone that is maximal in the countable-Borel lattice, by a corollary of the Pettis theorem [1, Theorems 9.9,9.10], every Borel-affine function is necessarily continuous and therefore essentially-finite. Hence, every subclone B of the affine clone A in Post's lattice, has only one clone in the countable-Borel lattice that restricts to it, namely $\langle B\rangle^{<\omega_1}$.

The rest of Post's lattice can be categorized into two sections. The 'top' of the lattice, which consists of intersections of the maximal clones of the lattice, and the 'sides' of the lattice, where only one 'side' needs to be considered due to the De Morgan dual. For both of these sections, the following two functions are of great relevance

$$\nabla^\omega(\vec{x}) \coloneqq x_0 \wedge (x_1 \vee x_2 \vee \ldots) \qquad \Delta^\omega(\vec{x}) \coloneqq x_0 \vee (x_1 \wedge x_2 \wedge \ldots) = \delta(\nabla).$$

3.3 The Sides of Post's Lattice

Whereas the maximal clones in the countable-Borel lattice are clearly defined as polymorphism clones, the sides aren't as clear. This complication in terms of the view of relations perhaps lends to a more complex structure than the top of the lattice. This is best illustrated through the countable-Borel extensions of $\langle\wedge\rangle$ and $\langle\vee\rangle$.

Clones that Restrict to $\langle\wedge\rangle$ and $\langle\vee\rangle$

In Post's lattice, these clones, being towards the bottom, are some of the simplest. For example, in $\langle\wedge\rangle$, all functions within the clone take the form of $x_{i_1} \wedge \ldots \wedge x_{i_n}$. However, when extending the clone into the countable-Borel setting, Lemma 3.1 leads to pathological behavior. In fact, we predict that there may be uncountably many clones K that restrict to $\langle\wedge\rangle$. For example, there are the natural extensions of $\langle\wedge\rangle^{<\omega}$: $\langle\wedge\rangle^{<\omega_1}$ and $\langle\bigwedge\rangle^{<\omega_1}$, but also clones such as $\langle \liminf\rangle^{<\omega_1}$, where

$$\liminf{}^\omega(\vec{x}) \coloneqq \bigvee_{\substack{S \subseteq \mathbb{N} \\ S \text{ is finite}}} \left(\bigwedge_{j \in \mathbb{N} \setminus S} x_j \right).$$

The lower sides, $\langle \wedge \rangle, \langle \wedge, 0(\vec{x}) \rangle, \langle \wedge, 1(\vec{x}) \rangle, \langle \wedge, 0(\vec{x}), 1(\vec{x}) \rangle$, and their duals, are still under investigation.

Clones that Restrict to $T_{0,\infty}$ and $T_{1,\infty}$

In fig. 1, for finite n,

$$f \in T_{0,n}^{<\omega} \text{ iff } \left(\forall (\vec{x}_i)_{i<n}, \bigwedge_{i=0}^{n-1} \vec{x}_i = \vec{0} \text{ implies that } \bigwedge_{i=0}^{n-1} f(\vec{x}_i) = 0 \right),$$

and $f \in T_{0,\infty}^{<\omega}$ if and only if $\forall n \in \mathbb{N}, f \in T_{0,n}^{<\omega}$. Of particular relevance to other clones is the countable-Borel clone of $T_{0,\omega}$ which restricts to $T_{0,\infty}^{<\omega}$. In this clone,

$$f \in T_{0,\omega} \text{ iff } \left(\forall (\vec{x}_n)_{n\in\mathbb{N}}, \bigwedge_{n\in\mathbb{N}} \vec{x}_n = \vec{0} \text{ implies that } \bigwedge_{n\in\mathbb{N}} f(\vec{x}_n) = 0 \right) \text{ iff } \exists i \in \mathbb{N} : f(\vec{x}) \leq x_i.$$

Using the second characterization of $T_{0,\omega}$, one can see that $T_{0,\omega} = \langle \not\Rightarrow, \nabla \rangle^{<\omega_1}$. An alternative proof takes the approach of Section 4.3, by considering the operator

$$\gamma(f(x_1, ...)) \coloneqq x_0 \wedge f(x_1, ...).$$

The latter approach also provides the intersection of $T_{0,\omega}$ with clones that restrict to $M^{<\omega}, T_1^{<\omega}$, and $MT_1^{<\omega}$ by mapping the generating sets under γ.

There are at least two clones in the countable-Borel lattice which restrict to each $T_{0,n}^{<\omega}$ and their variants; this section is still under investigation, however.

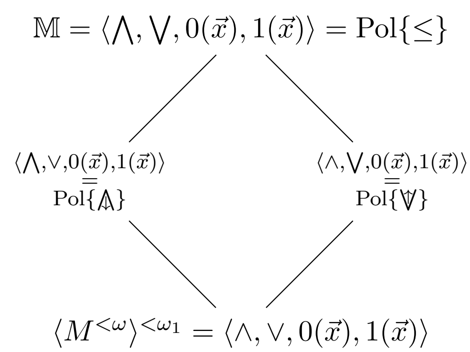
4 The Top of Post's Lattice

This section focuses on the top of the countable Post's Lattice, which consists of the sublattices of the clones that restrict to the maximal clones of $\mathbb{P}_2^{<\omega}$ and their intersections. We omit the superscript for Pol and $\langle - \rangle$ as we are solely working within the countable-Borel lattice.

4.1 Monotone Functions

Firstly, $\langle M^{<\omega} \rangle^{<\omega_1} = \langle \wedge, \vee, 0(\vec{x}), 1(\vec{x}) \rangle$, and $\mathbb{M} = \text{Pol}\{\leq\} = \langle \bigwedge, \bigvee, 0(\vec{x}), 1(\vec{x}) \rangle$ is the maximal clone of monotone-Borel functions. The generating set of $\mathbb{M}$ is known by a function being monotone if and only if it is positive [1, Theorem 28.11]. There are four clones which restrict to $M^{<\omega}$ in the manner of fig. 3.

Figure 3: Countable Borel Clones which Restrict to $M^{<\omega}$



Functions which preserve $\mathbb{A}$ preserve the convergence of decreasing sequences. In other words, if $\lim(\vec{x}_n) = \vec{x}_\infty$ where $(\vec{x}_n)$ is a decreasing sequence, then $\lim_{n\to\infty}(f(\vec{x}_n)) = f(\vec{x}_\infty)$. Analogously, functions which preserve $\mathbb{V}$ preserve the convergence of increasing sequences.

In the sublattice of clones that restrict to $M^{<\omega}$, $\langle\bigwedge,\vee,0(\vec{x}),1(\vec{x})\rangle$ and $\langle\wedge,\bigvee,0(\vec{x}),1(\vec{x})\rangle$ are the smallest properly-infinite clones. In other words,

Lemma 4.1. *Let $f^\omega\in\mathbb{M}$ be a discontinuous, monotone function. Then $\langle\bigwedge,\vee,0(\vec{x}),1(\vec{x})\rangle$ or $\langle\wedge,\bigvee,0(\vec{x}),1(\vec{x})\rangle\subseteq\langle\wedge,\vee,0(\vec{x}),1(\vec{x}),f\rangle$.*

Proof. Similar to Theorem 3.1, if $f^{-1}(0)$ is not closed, there exists a monotonically increasing sequence $(\vec{x}_n)_{n\in\mathbb{N}}\subseteq f^{-1}(0)$ converging to $\vec{x}_\infty\in f^{-1}(1)$. Thus, there exists a continuous, monotonically increasing $g:2^\omega\to2^\omega$ that maps each $1^n0\mapsto\vec{x}_n$ and $\vec{1}\mapsto\vec{x}_\infty$.

Since $\pi_n\circ g\in M^{<\omega}$, $\bigwedge=f\circ g\in\langle\wedge,\vee,0(\vec{x}),1(\vec{x}),f\rangle$.

If $f^{-1}(1)$ is not closed, we can analogously create a decreasing sequence that converges in $f^{-1}(0)$ and have $\bigvee\in\langle\wedge,\vee,0(\vec{x}),1(\vec{x}),f\rangle$. □

Also, $\langle\bigwedge,\vee,0(\vec{x}),1(\vec{x})\rangle$ and $\langle\wedge,\bigvee,0(\vec{x}),1(\vec{x})\rangle$ are the maximal clones with respect to $\mathbb{M}$ as $f\in\langle\bigwedge,\vee,0(\vec{x}),1(\vec{x})\rangle$ for all monotonic f where $f^{-1}(0)$ is open (resp. $f\in\langle\wedge,\bigvee,0(\vec{x}),1(\vec{x})\rangle$ for all monotonic f where $f^{-1}(1)$ is open). Since $\langle\bigwedge,\vee,0(\vec{x}),1(\vec{x})\rangle$ and $\langle\wedge,\bigvee,0(\vec{x}),1(\vec{x})\rangle$ are both the smallest and the maximal properly-infinite subclones of $\mathbb{M}$, fig. 3 is indeed the sublattice of clones that restrict to $M^{<\omega}$.

4.2 0-Preserving and 1-Preserving Functions

All of our discussion of $\vec{0}$-preserving functions can be applied to $\vec{1}$-preserving functions by the De Morgan dual. There are three clones that restrict to $T_0^{<\omega}$, which restrict in the manner of fig. 4.

Figure 4: Countable Borel Clones which Restrict to $T_0{}^{<\omega}$

$$\mathbb{T}_0=\langle\not\Rightarrow,\bigvee\rangle=\mathrm{Pol}\{\vec{0}\}$$

$$\langle\not\Rightarrow,\vee,\bigwedge\rangle=\mathrm{Pol}\{\lim=\vec{0}\}$$

$$\langle T_0^{<\omega}\rangle^{<\omega_1}=\langle\not\Rightarrow,\vee\rangle$$

A function preserves $\lim=\vec{0}$ if for all sequences $(\vec{x}_n)$ that converge to $\vec{0}$, $\lim\limits_{n\to\infty}(f(\vec{x}_n))=0$. In other words, f is continuous at $\vec{0}$.

Firstly, the smallest properly-infinite clone that restricts to $T_0^{<\omega}$ is $\langle\not\Rightarrow,\vee,\bigwedge\rangle$.

Lemma 4.2. *For every discontinuous, 0-preserving function $f\in\mathbb{T}_0$, $\bigwedge\in\langle\not\Rightarrow,\vee,f\rangle$.*

Proof. If $f^{-1}(0)$ is not closed, an adaptation of Theorem 3.1 suffices.

If $f^{-1}(1)$ is not closed, fix a boundary point y outside of $f^{-1}(1)$. Since Cantor space is totally-disconnected, there exists two open sets $A\ni y$ and $B\ni\vec{0}$ where $A\sqcup B=2^\omega$. Consider the clopen set $C\subset B$ containing $\vec{0}$ corresponding to $g^{-1}(0)$ for $g\in T_0^{<\omega}$. Then, $(g\wedge\neg f)^{-1}(0)$ is not closed and so Theorem 3.1 applies. □

Note that $\nabla(\vec{x})\in\langle\not\Rightarrow,\vee,\bigwedge\rangle$. This can be used to prove that if $\vec{0}\in\mathrm{Int}(f^{-1}(0))$ for $f\in\mathbb{T}_0$, then $f\in\langle\not\Rightarrow,\vee,\bigwedge\rangle$.

If $f^\omega\notin\langle\not\Rightarrow,\vee,\bigwedge\rangle$, then $\vec{0}$ is in the boundary of $f^{-1}(0)$ and so $\bigvee\in\langle\not\Rightarrow,f\rangle$. Through the characterization of Borel functions in Cantor space, $\mathbb{T}_0=\langle\not\Rightarrow,\bigvee\rangle$. Thus, there are only three clones that restrict to $T_0^{<\omega}$, as demonstrated by fig. 4.

4.3 Self-Dual Functions

In [2, Theorem 3.2.3.2], the self-dual functions of Post's lattice are investigated through two operators. Firstly, for $f^{n+1}(x_0, ..., x_n)$,

$$\alpha(f)(x_1, \ldots, x_n) \coloneqq f(0, x_1, \ldots, x_n).$$

Next, for $f^n(x_1, ..., x_n)$,

$$\beta(f)(x_1, \ldots, x_n) \coloneqq (\neg x_0 \wedge f(x_1, \ldots, x_n)) \vee (x_0 \wedge \neg f(\neg x_1, \ldots, \neg x_n)) = x_0 \ ?\ f(x_1, \ldots, x_n) : \neg f(\neg x_1, \ldots, \neg x_n).$$

For any clone F not containing constant functions, $\langle F \cup \{0\}\rangle = \alpha(F)$, so $\alpha(F)$ is a clone.

Recall that $D^{<\omega}$ is the clone of self-dual functions in Post's lattice. For any function $f \in \mathbb{P}_2^{<\omega}$,

- $\alpha(\beta(f)) = f$,
- $\beta(f) \in D^{<\omega}$,
- $\beta(\alpha(f)) = f \iff f \in D^{<\omega}$.

Thus, α, β give a bijection

$$\mathbb{P}_2 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} D.$$

For any subclone $F \subseteq D^{<\omega}, \beta(\alpha(F)) = F \subseteq \alpha(F)$. For any $G \subseteq \mathbb{P}_2^{<\omega}$, $\alpha(\beta(G)) = G$. If G is a clone, $0(\vec{x}) \in G$, and $\beta(G) \subseteq G$ (which immediately holds from $G = \alpha(F)$), then $\beta(G) = D \cap G \subseteq D$ is a clone.

We have an order-isomorphism

$$\{G \in \mathrm{Clo}^{<\omega}(2) \text{ s.t. } 0 \in G \text{ and } \beta(G) \subseteq G\} \underset{\beta = D^{<\omega} \cap (-)}{\overset{\alpha}{\rightleftarrows}} \{G \in \mathrm{Clo}^{<\omega}(2) \text{ s.t. } G \subseteq D\}.$$

For any subset $G \subseteq \mathbb{P}_2^{<\omega}$, $\beta(\langle G\rangle) \subseteq \langle \beta(G)\rangle$. Also, $\beta(\langle G\rangle) \subseteq \langle G\rangle$ if and only if $\beta(G) \subseteq \langle G\rangle$.

In Post's lattice, the clones which are closed under β, contain $0(\vec{x})$ and are not below $A^{<\omega}$ are $MT_{0,2}^{<\omega}, T_0^{<\omega}$, and $\mathbb{P}_2^{<\omega}$. These respectively correspond to the three self-dual clones not below $A^{<\omega}$; $DM^{<\omega}, DT_0^{<\omega} = DT_1^{<\omega} = DT_0T_1^{<\omega}$, and $D^{<\omega}$.

The operators α and β extend to countable-Borel clones $K \subseteq \mathbb{B}$, where for f^ω,

$$\alpha(f)(x_0, \ldots) \coloneqq f(0, x_0, \ldots) \text{ and } \beta(f)(x_0, \ldots) \coloneqq x_0 \ ?\ f(x_1, \ldots) : \neg f(\neg x_1, \ldots).$$

These operators satisfy the above properties in the countable-Borel setting. Thus, there are two countable-Borel clones which restrict to $D^{<\omega}$, namely

- $\langle D^{<\omega}\rangle^{<\omega_1} = \langle \beta(\wedge), \beta(\neg)\rangle = \langle \between_2^3, \neg\rangle$,
- $\mathbb{D} = \mathrm{Pol}\{\neg\} = \langle \beta(\bigwedge), \beta(\neg)\rangle = \langle \bigwedge(\neg x_0, x_1, \ldots) \vee \nabla(\vec{x}), \neg\rangle$,

where $\between_2^3(\vec{x}) \coloneqq (x_1 \wedge x_2) \vee (x_1 \wedge x_3) \vee (x_2 \wedge x_3)$. As in the case of Post's lattice, the clones which restrict to $DM^{<\omega}$ and $DT_0T_1^{<\omega}$ follow from the same argument and will be discussed in the following subsection.

4.4 Intersections of Maximal-Post Clones

This subsection focuses on clones which restrict to intersections of maximal clones in Post's lattice.

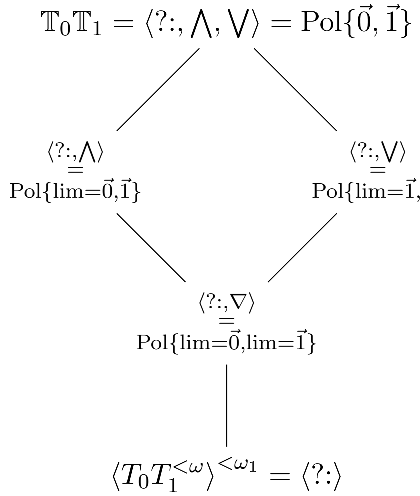
Firstly, the intersection of an essentially-finite clone K and any clone I equals $\langle K \cap I \cap \mathbb{P}_2^{<\omega}\rangle^{<\omega_1}$. Thus, our interest is solely in intersections of properly-infinite clones. Note in the following illustrations that each countable-Borel clone that restricts to an intersection-Post clone $X^{<\omega} \cap Y^{<\omega}$ is the intersection of two clones that each restrict to $X^{<\omega}$ and $Y^{<\omega}$.

Once again, not depicted in the following figures, for each of the following clones in the sublattices that restrict to $M^{<\omega}, T_1^{<\omega}$ and $MT_1^{<\omega}$, their intersections with $T_{0,\omega}$ are described by mapping the generating functions under γ.

T_0T_1 Clones

Since the properly-infinite clones in the T_0 and T_1 sublattices depended on whether or not $\vec{0}$ or $\vec{1}$ were in the interiors or boundaries of their respective sets, it is natural to approach this sublattice in a similar vein. Thus, the subclones of $\mathbb{T}_0\mathbb{T}_1$ end up conditioning on whether or not $\lim = \vec{0}$ or $\lim = \vec{1}$ are preserved. Indeed, there are five clones that restrict to $T_0T_1^{<\omega}$ in the manner of fig. 5.

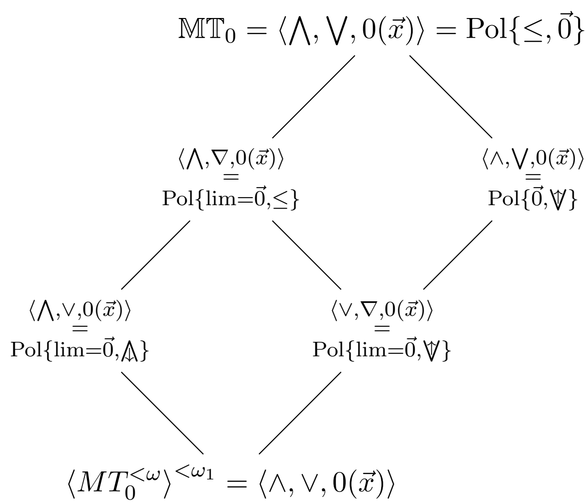
Figure 5: Countable Borel Clones which Restrict to $T_0T_1{}^{<\omega}$



MT_0 and MT_1 Clones

Again, the following discussion of monotone $\vec{0}$-preserving functions applies to monotone $\vec{1}$-preserving functions by the De Morgan dual. There are six clones that restrict to $MT_0^{<\omega}$, forming a sublattice as in fig. 6.

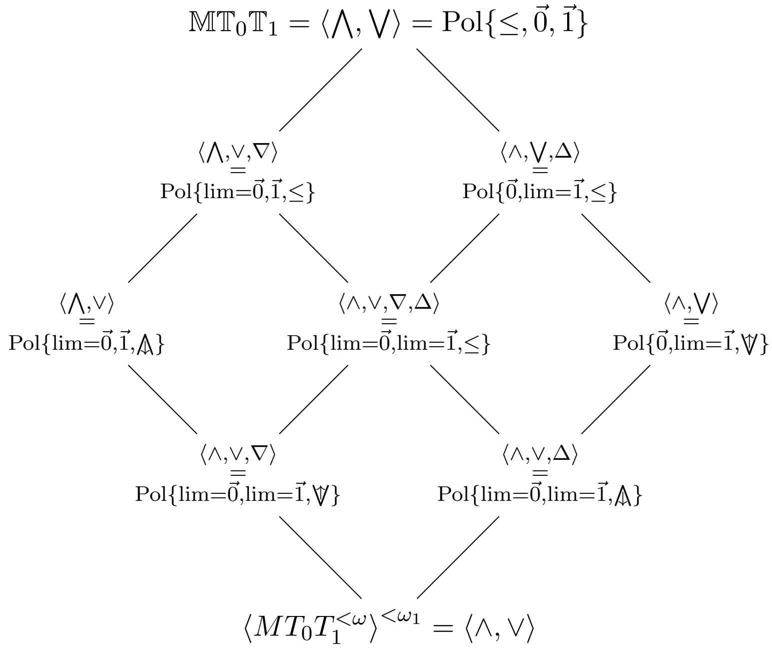
Figure 6: Countable Borel Clones which Restrict to $MT_0{}^{<\omega}$



MT_0T_1 Clones

There are nine clones that restrict to $MT_0T_1^{<\omega}$, illustrated in fig. 7.

Figure 7: Countable Borel Clones which Restrict to $MT_0{T_1}^{<\omega}$



DT_0T_1 Clones

As discussed in the previous subsection, $DT_0 = DT_1 = DT_0T_1$. Thus, there are three clones that restrict to $DT_0T_1^{<\omega}$, given by the images of the generating sets of clones which restrict to $T_0^{<\omega}$ under β. These clones are namely

- $\langle DT_0T_1^{<\omega} \rangle^{<\omega_1} = \left\langle \Diamond_2^3, +^3 \right\rangle$,
- $\left\langle \Diamond_2^3, +_3, \beta(\bigwedge) \right\rangle = \mathrm{Pol}\{\neg, \lim = \vec{0}, \lim = \vec{1}\} = \left\langle \Diamond_2^3, +^3, \bigwedge(\neg x_0, x_1, \ldots) \vee \nabla(\vec{x}) \right\rangle$,
- $\mathbb{DT}_0\mathbb{T}_1 = \mathrm{Pol}\{\neg, \vec{0}, \vec{1}\} = \left\langle \Diamond_2^3, +^3, \beta(\bigvee) \right\rangle = \left\langle \Diamond_2^3, +^3, \nabla(\neg x_0, x_1, \ldots) \vee \bigwedge(\vec{x}) \right\rangle$,

forming a sublattice analogous to fig. 4. Note that $+^3(\vec{x}) \coloneqq x_0 + x_1 + x_2$, where $+$ is the exclusive-or operation.

DM Clones

Since the clones which restrict to $DM^{<\omega}$ are images of the clones that restrict to $MT_{0,2}^{<\omega}$, we require knowledge of said clones. As discussed, they are not known and are being investigated.

References

[1] A. Kechris, "Classical Descriptive Set Theory," Springer-Verlag, 1995

[2] D. Lau, "Function Algebras on Finite Sets, A Basic Course in Multi-Valued Logic and Clone Theory," Springer, 2006