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Survey:

Explorations in Theorems of Schur, Schur-Horn For Hermitian Matrices

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-Explorations in Theorems of Schur, Schur-Horn for Hermitian matrices-

1. We see explicitly in the papers of Bhatia [2001], Professor Bloch [1990] applications of the Schur-Horn theorem using Hermitian matrices.
Note: a Hermitian matrix $H = H^*$ can be written as $\bar{H}^T$, where $\bar{H}$ denotes the complex conjugate of complex entries in our matrix and $\bar{H}^T$ indicates taking the transpose of the resulting matrix.

Schur-Horn Theorem - for two sequences $d_1, d_2, \ldots, d_n$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ arranged $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ (non-increasing order), $\exists$ Hermitian matrix H with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and diagonal entries $d_1, d_2, \ldots, d_n$ if and only if

$$d_1 \leq \lambda_1,$$

$$d_1 + d_2 \leq \lambda_1 + \lambda_2,$$

$$\vdots$$

$$d_1 + d_2 + \ldots + d_n = \lambda_1 + \lambda_2 + \ldots + \lambda_n$$

Essentially, this theorem states that for a Hermitian matrix H, we can order diagonal entries and eigenvalues such that we can compare diagonal entries and their sums to eigenvalues and their sums. We have numerical equality at endpoints (final terms) for $\boxed{\sum_{i=1}^{n} d_i = \sum_{i=1}^{n} \lambda_i}$ (1) of real numbers.

As these papers progressed, these comparisons were also made between Hermitian matrices. Oftenly referenced were variations of the following trace identity $\boxed{\sum_1 \gamma_i = \sum_1 \alpha_i + \sum_1 \beta_i}$ (2), which specifies eigenvalues of Hermitian matrices, $A, B, C = A+B$ as α_i, β_i and $\gamma_i = \alpha_i + \beta_i$ respectively, to also be arranged in non-increasing order i.e. $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$, $\beta_1 \geq \beta_2 \geq \ldots \geq \beta_n$ and $\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_n$.

A For sums of varyingly indexed eigenvalues, we have these following Weyl's inequalities characterizing the eigenvalues in Bhatia[2] as: $\boxed{\gamma_{i+j-1} \leq \alpha_i + \beta_j}$ for $i+j-1 \leq n$

(3)

Ex. 1 Let's look more closely at an example of λ_i fulfilling these inequalities. Let there be a Hermitian matrix $H = \begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix}$.
To find eigenvalues that will correspond to the diagonal entries of this matrix, we use $\det(H - \lambda I)$ to find roots of a resulting characteristic polynomial. Note I is identity matrix in $\mathbb{R}^{2\times 2}$ i.e. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
So we then have $\det\left(\begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$

$$\det\left(\begin{bmatrix} 1-\lambda & 1+i \\ 1-i & 2-\lambda \end{bmatrix}\right) = (1-\lambda)(2-\lambda) - (1+i)(1-i) = 0$$

$$= (2 - 3\lambda + \lambda^2) - (1 - i^2), \quad i = \sqrt{-1}$$

$$= 2 - 3\lambda + \lambda^2 - 1 + i^2 = 0$$

$$= 2 - 3\lambda + \lambda^2 - 1 - 1 = 0$$

$$= 2 - 3\lambda + \lambda^2 - 2 = 0$$

$$= \lambda^2 - 3\lambda = 0$$

$$= \lambda(\lambda - 3) = 0$$

$$\therefore \lambda_2 = 0, \lambda_1 = 3$$

Now looking back at Hermitian H and Schur-Horn inequalities, we arrange λ_1, λ_2 as $\lambda_1 \geq \lambda_2$. So $H = \begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix}$ has

diagonal entries $d_1 = 1$, $d_2 = 2$. Note $\lambda_1 = 3 \geq 1 = d_1$, and that $d_1 + d_2 = 1 + 2 = 3 + 0 = \lambda_1 + \lambda_2$.
And since $d_1 \leq \lambda_1$ but $d_1 + d_2 = \lambda_1 + \lambda_2$, then Schur-Horn's inequalities hold true.

Before making a remark about Ex. I, let's look at the concept of majorisation. Let column vectors $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$, $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n$ be organised such that

$\vec{x}_j^{\downarrow}$ denotes rearranging this vector's coordinates in decreasing order as described in Bhatia[2]. Then x is majorised by y i.e. $x \prec y$, if both $\sum_{j=1}^{k} x_j^{\downarrow} \leq \sum_{j=1}^{k} y_j^{\downarrow}$, $1 \leq k \leq n$

and $\sum_{j=1}^{n} x_j^{\downarrow} = \sum_{j=1}^{n} y_j^{\downarrow}$. (4)

This is particularly useful, since shown more succinctly in Ex. I that we have the majorisation $d \prec \lambda$, analogous to $d \prec \alpha$ in Bhatia[2] for diagonal entries to particular Hermitian matrix H.

Also important are Ky Fan's inequalities

$$\boxed{\sum_{j=1}^{k} \gamma_j \leq \sum_{j=1}^{k} \alpha_j + \sum_{j=1}^{k} \beta_j, \quad 1 \leq k \leq n.} \quad (5)$$

Note that it was noted in Bhatia[2] that when $k=n$, we have our trace identity $\sum \gamma_j = \sum \alpha_j + \sum \beta_j$.

What is notable from these inequalities is that they can be plotted to create convex shapes known as polytopes in the case of Weyl's original family of inequalities. We can see this based in an example similar to the one given in Section 3 of Bhatia[2].

From $n=2$ ($\mathbb{R}^{2\times 2}$) case, we see in Bhatia that the Weyls inequalities give

$$\boxed{\gamma_1 \le \alpha_1 + \beta_1 \; , \; \gamma_2 \le \alpha_1 + \beta_2 \; , \; \gamma_2 \le \alpha_2 + \beta_1 .} \quad ⑤$$

Along with the trace inequality, these inequalities are enough to characterise eigenvalues α, β, γ relative to Hermitian matrices $A, B, C = A+B$ respectively. Given $\alpha_n, \beta_n, \gamma_n$ above for $n=2$, we then know from the Schur-Horn theorem that these eigenvalues again correspond to existence of Hermitian matrices A, B, C.

Ex 2. Choose pairs for α, β as

$$\alpha_1 = 5, \; \alpha_2 = 2 \quad \text{and} \quad \beta_1 = 4, \; \beta_2 = -1$$

From trace identity, we know

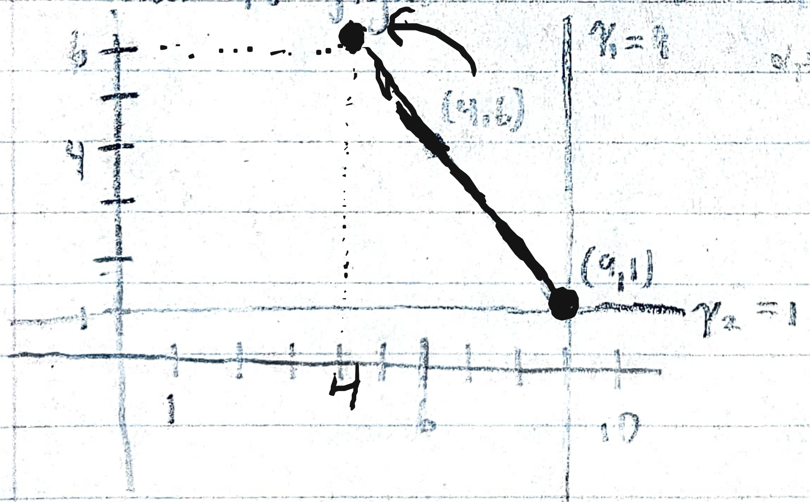
$$\begin{aligned} \gamma_1 + \gamma_2 &= (\alpha_1 + \alpha_2) + (\beta_1 + \beta_2) \\ \| \quad\; \| \\ \gamma_1 + \gamma_2 &= (\alpha_1 + \beta_1) + (\alpha_2 + \beta_2) \\ \| \quad\; \| \\ 10 = 9 + 1 &= (9) + (1). \end{aligned}$$

$\gamma_1 + \gamma_2 = 9 + 1 = 10$ plots a line in $\mathbb{R}^2$,
but $\gamma_1 \geq \gamma_2$ restricts this line.

Then from Weyl's inequalities, we have

$\gamma_1 \leq (\alpha_1 + \beta_1) =$ $\quad$ $\gamma_2 \leq (\alpha_1 + \beta_2) =$; $\gamma_1 \leq 9$
$\gamma_1 \leq (5+4) = 9$, $\gamma_2 \leq (5-1) = 4$; $\gamma_2 \leq 4$

Fig. 1 * Plotting these inequalities looks like this figure similar to figure 1 in Bhatia section 3.



$\alpha_1+\beta_1$, $\alpha_2+\beta_2$
(9,1) $\quad$ (4,6)
result of $C_{\frac{\pi}{2}}$

This is indeed a line segment from Weyl's inequalities.

It is important to show each point on line segment corresponds to two eigenvalues of Hermitian matrix $C = A + B$, given $\alpha(A) = (5,2)$, $\beta(B) = (4,-1)$.

We construct diagonal matrices

$$A = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}.$$

Let $U_\theta \in \mathbb{R}^{2\times2}$ be the rotation matrix

$$U_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

also let

$B_\theta = U_\theta B_0 U_\theta^*$, $C = A + B_\theta$

Thus giving family of Hermitian matrices parametrised by θ; here, we evaluate at $\theta = \frac{\pi}{2}$

Note: $C_0 = A + B_0 = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}$

So $C_{\pi/2} = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}$,

Which is the second endpoint on our line of eigenvalues. (See Fig. 1)

We finally and conclusively return to the topic on p. 5 of plotting convex shapes, namely convex polytopes.

To demonstrate this, we utilize permutation matrices, which are a rearrangement of matrix rows $n!$ times.

Ex 3. For example, if we permute the identity matrix rows in $\mathbb{R}^{2\times 2}$, we have: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (No change), $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

These two matrices are thus representative of a convex set, hence $\mathbb{R}^{2\times 2}$ has $2!$ or 2 elements in the convex set.

Ex 4. I was shown by Professor Bloch the tetrahedron in $\mathbb{R}^{3\times 3}$, shown in Figure 2 below. Graphic:

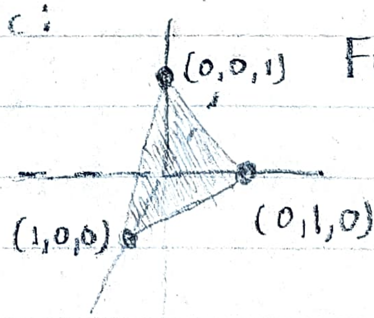


Fig. 2

One remarkable thing to note is the effect of permuting the identity matrix in $\mathbb{R}^{3\times 3}$. Considering $3!$ permutations, namely 6 of them, we only see 3 vertices. So albeit 6 elements represent this convex set, 3 matrices provide 3 redundant vertices. So we plot 3 points in this case. These shapes that we can acquire due to these $n!$ permutations are called permutohedrons.

Along with the previous remark in mind, we can observe the Schur Horn inequalities geometrically. As discussed with Professor Bloch, we can see what happens when we permute eigenvalue entries. As stated succinctly in Section 4 in Professor Bloch's

Ex. 5 paper, we can view $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, and $a = (a_1, \ldots, a_n)$, the eigenvalues and diagonal entries of hermitian matrix H, as vectors in $\mathbb{R}^n$.

Fig. 3

(a_1, a_2, a_3)

When we permute $(\lambda_1, \lambda_2, \lambda_3)$ for some hermitian matrix H, we get the permutohedron shown in Fig. 3.

Since the Schur-Horn theorem has two implications, then let's see them in relation to this convex polytope.

$\Rightarrow$ Schur says in this case: if $\exists$ Hermitian matrix with diagonal entries (a_1, a_2, a_3), eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$, then vector $(a_1, a_2, a_3) = \vec{a}$ lies inside the permutohedron generated by $3!$ permutations of $(\lambda_1, \lambda_2, \lambda_3)$.

$\Leftarrow$ Horn says in this case: if vector $(a_1, a_2, a_3) = \vec{a}$ lies inside the permutohedron generated by $3!$ permutations of $(\lambda_1, \lambda_2, \lambda_3)$, then $\exists$ Hermitian matrix with diagonal entries (a_1, a_2, a_3) and eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$.

To conclude, these two statements geometrically verify existence of a Hermitian matrix, with diagonal entries $(a_1, \ldots, a_n)$ that have corresponding eigenvalues $(\lambda_1, \ldots, \lambda_n)$, which is illustrated by the convex polytope.

Acknowledgements

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References

[1] Professor Anthony M. Bloch, Steepest Descent, Linear Programming, and Hamiltonian Flows, American Mathematical Society 114, (1990)

[2] Rajendra Bhatia, Linear Algebra to Quantum Cohomology: The Story of Alfred Horn's Inequalities, The Mathematical Association of America, 108, (2001)