THE GROMOV BOUNDARY OF HYPERBOLIC GROUPS AND FINITE STATE AUTOMATA

ABHI SHUKUL

Abstract. We investigate a connection between the geometric and computational properties of groups with "coarse negative curvature". In particular, we use the existence of an automatic structure on hyperbolic groups to characterize when the "boundary at infinity" is finite, or equivalently, when the corresponding hyperbolic group is virtually cyclic.

1. Introduction

1.1. **Overview.** For a finitely presented group, we can construct a corresponding geometrical object called a *Cayley graph*, which we can endow with a metric. In this paper, we consider a special class of groups called *hyperbolic groups*, which are finitely presentable groups whose corresponding Cayley graph metric has "coarse negative curvature". For groups with this property, we can define a corresponding space called the *Gromov boundary*, which, roughly speaking, can be thought of as the "boundary at infinity" of the corresponding Cayley graph.

It turns out that hyperbolic groups have some nice computational properties. In particular, these groups have an *automatic structure*, which means there exist finite state automata that solve certain decision problems about the group. The automatic structure is powerful in that, among other things, it can give us a solution to the word problem, a problem that is undecidable for arbitrary finitely presented groups[1].

Let G be a finitely presented hyperbolic group with generating set A. Let Γ be the corresponding Cayley graph, $\partial\Gamma$ be the Gromov boundary, and let W be the *word acceptor* automaton (this exists as part of the automatic structure). The main result of this paper is the following:

Theorem 1.1. *$\partial\Gamma$ is finite if and only if no distinct cycles in W share a state.*

Now, recall that a group is *virtually cyclic* if it contains a finite-index cyclic subgroup. With this notion, we have the following consequence of Theorem 1.1:

Corollary 1.2. *G is virtually cyclic if and only if no distinct cycles in W share a state.*

In order to see the geometric aspect of this, we note that G being infinite and virtually cyclic is equivalent to saying that the Cayley graph of G is *quasi-isometric* to the Cayley graph of $\mathbb{Z}$, which means there's a map between the two Cayley graphs (which are metric spaces) that preserves distance up to some bounded error[2]. In order to see the forward direction, we can consider the natural group action of the finite index cyclic subgroup on the

Date: August 25, 2023.

[1] See [4, Section 2.3]

[2] For a complete definition of quasi-isometries, see [2, Ch. 7]

Cayley graph of G and apply the Švarc-Milnor lemma[3]. The reverse direction follows from Theorem 7.6 of [2].

1.2. **Acknowledgements.** This research was conducted as part of the 2023 University of Michigan Mathematics REU program. The author would like to thank his mentor Teddy Weisman for his invaluable guidance, patience, and warmth. Without his advice, this project wouldn't have been possible.

2. Background

2.1. **Group Presentations.** In order to see the computational side of groups, we must think about groups from the perspective of languages. In this section, we recall the definition of a group presentation. We will loosely follow the description given in [2, 1.4]. The reader may also consult [4, 2.1] for a more in-depth investigation.

First, let $A = \{a_1, \dots, a_n\}$ be an alphabet and let $A^{-1} = \{a_1^{-1}, \dots, a_n^{-1}\}$. A *word* over A is a finite string made up of letters in $A \cup A^{-1}$. We can take any word over A and reduce it by removing all instances of $x_i x_i^{-1}$ and $x_i^{-1} x_i$. A word without such instances is called a *reduced word*. We can define the *free group* $F(A)$ to be the set of reduced words over A with the binary operation being concatenation followed by reduction. One can check that this forms a group by noting that the identity is the empty word, which we will denote as ε for the rest of this paper.

Definition 2.1. A *group presentation* is a pair (A, R) where A is an alphabet and R is a set of reduced words over A. If we let H be the smallest normal subgroup containing R, then

$$\langle A \mid R \rangle := F(A)/H.$$

A group presentation (A, R) is said to be finite if A and R are finite.

Definition 2.2. A group G is *finitely presentable* if there exists a finite group presentation (A, R) such that $G \cong \langle A \mid R \rangle$.

If G admits a group presentation $\langle A \mid R \rangle$, we say that A is a *generating set* of G.

2.2. **Hyperbolic Groups and the Gromov Boundary.** Let M be a metric space. A *geodesic segment* $\gamma : [a, b] \to M$ is an isometric embedding with $a \leq b$, and a *geodesic ray* is an isometric embedding of $[0, \infty)$. M is said to be *geodesic* if for any $x, y \in M$, there exists a geodesic segment $\gamma : [a, b] \to M$ such that $\gamma(a) = x$ and $\gamma(b) = y$.

For a metric space M and a set $A \subseteq M$, $\mathcal{N}_r(A) := \bigcup_{x \in A} B_r(x)$. We say that $\mathcal{N}_r(A)$ is the r-neighborhood of A.

Definition 2.3. A geodesic metric space M is said to be δ-*hyperbolic* provided that for any geodesic triangle with sides (geodesic segments) α, β, γ, we have $\alpha \subseteq \mathcal{N}_\delta(\beta) \cup \mathcal{N}_\delta(\gamma)$. We say that M is *hyperbolic* if there exists some $\delta > 0$ such that M is $\delta-$hyperbolic.

For a δ-hyperbolic metric space M, we can give a precise definition for the "boundary at infinity" of M:

Definition 2.4. Let $c_1, c_2 : [0, \infty) \to M$ be geodesic rays based at the identity ($c_1(0) = c_2(0) = \varepsilon$). We say that c_1, c_2 are *asymptotic* provided that $\sup_t(c_1(t), c_2(t)) < \infty$. This defines an equivalence relation on the set of geodesic rays based at the identity. The *Gromov boundary*, denoted ∂M, is the set of equivalence classes under this relation.

[3] See [1, I.8]

Now we'll describe how to associate a geometry to a finitely presentable group so we can think about δ-hyperbolicity and the Gromov boundary in the context of groups. Let G be a group with presentation $\langle A, R\rangle$.

Definition 2.5. The *Cayley graph* $\Gamma(G, A)$ is a directed, labelled graph where the vertex set V is G and for any vertices $g_1, g_2 \in G$, there is a directed edge from g_1 to g_2 labelled a provided that $g_1 a = g_2$ and $a \in A \cup A^{-1}$.

Remark. The Cayley graph depends on the choice of A, but if we have another generating set A', then $\Gamma(G, A)$ and $\Gamma(G, A')$ are quasi-isometric[4].

Let $d : G \times G \to \mathbb{R}$ such that for $g_1, g_2 \in \Gamma$, $d(g_1, g_2)$ equals the length of a shortest path between g_1, g_2 in Γ. We note that d is well-defined since our Cayley graph must be connected, and one can check that d is a metric on $\Gamma(G, A)$. Moreover, $\Gamma(G, A)$ gives us a geodesic metric space if we "attach" unit [0,1] intervals at each edge and identify the end points with the vertices the edge connects. We call this the *geometric realization* of $\Gamma(G, A)$, but for the rest of this paper, we'll simply refer to it as $\Gamma(G, A)$.

We can think of a finite word over A as a path in the Cayley graph starting at the origin and ending at the vertex corresponding to the group element represented by the word. In this case, a word w over A is geodesic if the corresponding path is a geodesic segment. This equivalent to saying that w is geodesic if its a shortest possible representative of its corresponding group element. For this paper, $\hat{w}$ will refer to the path in the Cayley graph while w will refer to the vertex corresponding to the group element it represents.

Definition 2.6. G with generating set A is *$\delta-$hyperbolic* provided that the metric space $\Gamma(G, A)$ is δ-hyperbolic.

Remark. Hyperbolicity is an invariant of G, but the choice of δ depends on the generating set A[5].

We note that geodesic rays only exist for $\Gamma(G, A)$ when G is infinite. In fact, $\partial\Gamma(G, A)$ is empty if and only if G is finite.

We will now describe a couple results about hyperbolic metric spaces and their boundaries that'll be useful for proving our main result and understanding why hyperbolic groups have nice computational properties.

Lemma 2.7 (Asymptotic Rays are Uniformly Close)**.** *Let M be a proper δ-hyperbolic metric space and let $c_1, c_2 : [0, \infty) \to M$ be geodesic rays based at the identity such that c_1 and c_2 are asymptotic. Then, for all $t > 0$, $d(c_1(t), c_2(t)) < 2\delta$*

Proof. See [1, III.H.3]

■

Lemma 2.8. *Let M be a δ-hyperbolic metric space. Then, $|\partial M|$ is either 0,2, or uncountably infinite*

Proof. See [3, 11.15] ■

[4] For more on quasi-isometries, see [2, Ch. 7]

[5] This follows quasi-isometries preserve hyperbolicity

2.3. **Finite State Automata.**

Before describing automatic structures, we will briefly recall the definition of a finite state automaton.

Definition 2.9. A *finite state automaton* is a 5-tuple (S, A, μ, Y, s_0) where S is a finite set of *states*, A is an alphabet, $\mu : S \times A \to S$ is the *transition function*, $Y \subseteq S$ is the set of *accept states*, and $s_0 \in S$ is the *starting state*.

The idea is that the FSA (finite state automaton) takes in a finite word over A and reads each letter one by one starting from the left-hand side. Our initial current state is s_0, and when we read the first letter $a_0 \in A$, we proceed to the state $s_{k_1} := \mu(s_0, a_0)$. Then, we read the next letter a_1 and proceed to the state $s_{k_2} := \mu(s_{k_1}, a_1)$. We continue doing this until the entire word has been read. Then, if the final state is in Y, the FSA *accepts* the word. Otherwise, we say it *rejects* the word. For an FSA F with alphabet A, $L(F)$ will denote the set of finite words over A that are accepted by F.

An FSA can equivalently be thought of as a directed, labelled graph. The elements of S correspond to the set of vertices, and for each $s \in S, a \in A$, there is a directed edge from s to $\mu(s, a)$ labelled 'a'. At each vertex, for each $a \in A$, there is at most one edge labeled 'a' going out of it. In this case, each finite word corresponds to a path starting at s_0, and words that are accepted correspond to directed paths whose last state is an accept state. We will use this perspective in this paper.

When utilizing the directed graph perspective, we can make a few simplifications that do not change the set of accepted words. First, we can remove states in S that cannot be reached from the start state. Second, we can remove all non-accept states from which there is no path to an accept state. This will involve omitting edges going to these states so that when running through our simplified FSA, if we read a letter and there is no corresponding edge from the current state, the word is rejected. This simplified automaton is called a *normalized finite state automaton*. For the rest of this paper, we will assume every FSA is normalized.

For more on finite state automata, the reader may consult [4].

2.4. **Automatic Structures.**

Let G be a finitely presented group with generating set A.

Definition 2.10. An *automatic structure* on G consists of the following finite state automata: the word acceptor automaton W over A and the multiplier automaton M_x over (A, A) for $x \in A \cup \{\varepsilon\}$. These automata satisfy the following properties:

(1) Every element of G is represented by a word in $L(W)$
(2) For $x \in A \cup \{\varepsilon\}$, $(w_1, w_2) \in L(M_x)$ if and only if $w_1 x = w_2$ and $w_1, w_2 \in L(W)$

We say that G is *strongly geodesically automatic* if there exists an automatic structure where $L(W)$ is the set of all geodesic words over the generating set.

For our main result, we won't be needing the multiplier automaton, so we'll only present the results necessary to show the existence of the word acceptor for hyperbolic groups. The reader may refer to [4] for the existence of the multiplier automaton for hyperbolic groups.

It turns out that we can not only show that a hyperbolic group has an automatic structure, but that it is strongly geodesically automatic. We will now present some results from [4] that will allow us to see this fact.

First, recall that for a metric space M and $X, Y \subseteq M$, the *hausdorff distance* between X and Y is $\inf\{r > 0 \mid A \subseteq \bigcup_{x \in A'} B_r(x) \text{ and } A' \subseteq \bigcup_{x \in A} B_r(x)\}$.

Theorem 2.11. *Let G be a finitely presented group with generating set A. Suppose there exists $k > 1$ such that for any two geodesic words v, w over A where $d(v, w) < 1$, the hausdorff distance between the paths $\hat{v}, \hat{w}$ is at most k. Then, it follows that G is strongly geodesically automatic.*

Proof. See [4, 3.2]. ■

This tells us that hyperbolic groups are strongly geodesically automatic since one can check that δ-hyperbolicity implies that the hypothesis of Theorem 2.11 is satisfied.

Now, it is desirable to be able to have a word acceptor automaton that accepts a unique geodesic word for each group element. We can do this by considering an ordering on our alphabet A. Then, we can consider a *shortlex* ordering on words over A, where for words v, w, $v < w$ if and only if either v is shorter than w, or if they're the same length, then v comes before w in lexicographical order (using the ordering on A). This defines a well-ordering, so for each group element, there exists a minimal geodesic word representing it, which we'll call a *shortlex geodesic word*. Now, if there exists a word acceptor automaton that accepts the language of shortlex geodesic words, then we say that the group is *Shortlex-automatic*.

Theorem 2.12. *A strongly geodesically automatic group is Shortlex-automatic for any ordering of the generators.*

Proof. See [4, 2.5]. ■

This result tells us that given a hyperbolic group, there exists a shortlex geodesic word acceptor FSA. For Theorem 1.1, we will assume that W only accepts shortlex geodesic words.

3. Characterizing Hyperbolic Groups With Finite Gromov Boundary

Now that we've gone over the necessary background on hyperbolic groups and the automatic structure, we're ready to start proving Theorem 1.1.

First, let G be a $\delta-$hyperbolic group and let A be a finite set of semi-group generators of G. Let $\Gamma(G, A)$ (which we'll denote Γ) be the corresponding Cayley graph of G with respect to A.

Definition 3.1. For a geodesic ray $c : [0, \infty) \to \gamma$, we define $f_c : \mathbb{Z}_{\geq 0} \to A$ to be

$$f_c(t) \begin{cases} c(0) & t = 0 \\ c(t-1)^{-1}c(t) & t > 0 \end{cases}.$$

Here $f_c(k)$ labels the kth edge in Γ of the geodesic ray $c : [0, \infty) \to \Gamma$.

Let $c_1, c_2 : [0, \infty) \to \Gamma$ be geodesic rays with the same base point such that they differ at at least one point. Then, there exists a t such that $c_1(t) = c_2(t)$ and $c_1(t + 1) \neq c_2(t + 1)$. We call any such t a *splitting point*. We call a geodesic ray $c : [0, \infty) \to \Gamma$ *shortlex* if each prefix if for every $t > 0$, the word $f_c(0)f_c(1) \dots f_c(t)$ is shortlex. For the rest of this paper, we'll assume that all geodesic rays have their base point at the identity element.

Definition 3.2. Let c_1, c_2 be shortlex geodesic rays. We can define an equivalence relation $\sim_E$ on shortlex geodesic rays as follows: $c_1 \sim_E c_2$ provided that there exist integers T_1, T_2 such that for all t, $f_{c_1}(T_1 + t) = f_{c_2}(T_2 + t)$. We can define an *end-behavior* to be an equivalence class of shortlex geodesic rays based the relation $\sim_E$.

Note that two shortlex geodesic rays can have at most 1 splitting point.

Lemma 3.3. *The shortlex geodesic rays with the same base point corresponding to the same boundary point $\overline{x} \in \partial\Gamma$ represent finitely many end behaviors.*

Proof. Suppose that there exist infinitely many shortlex asymptotic geodesic rays

$$c_1 : [0, \infty) \to \Gamma, c_2 : [0, \infty) \to \Gamma, c_3 : [0, \infty) \to \Gamma, \dots$$

such that for each $i \neq j$, c_i, c_j represent different end-behaviors. First, note that if $c, c' : [0, \infty) \to \Gamma$ are asymptotic geodesic rays, then for all t, $d(c(t), c'(t)) < 2\delta$. Second, for each t, $|B_\delta(c_1(t))| < C(\delta)$, where $C(\delta)$ is some constant that only depends on δ (this follows because each vertex in Γ has finite degree). Let $K > C(\delta)$ be a positive integer. For each $i, j \in \mathbb{N}$ with $i \neq j$, let t_{ij} be the splitting point of c_i, c_j. Let $A_K = \{(i, j) \mid i, j \in \{1, \dots, K\}, i \neq j\}$. Then, let $T = \max_{(i,j)\in A} t_{ij}$. Then, it follows that $c_1(T) \neq c_2(T) \neq \dots \neq c_K(T)$; however, for each $i \in \{1, \dots, K\}$, $c_i(T) \in B_\delta(c_1(T))$, giving us a contradiction because $|B_\delta(c_1(T))| < K$. ■

Let $W = S, A, \mu, Y, s_0)$ be the shortlex word acceptor automaton for Γ. Recall that an FSA can be represented as a labelled, directed graph. To make this more explicit, the set of vertices is S and the set of edges, which we'll call E_W, consists of edges (s_i, ℓ, s_j) where $\mu(s_i, \ell) = s_j$. Also, note that as mentioned earlier, we will assume that W is normalized.

Definition 3.4. A *simple closed path* $\mathcal{C}$ in W is a sequence of elements of E of the form

$$\mathcal{C} = ((s_{k_1}, \ell_{k_1}, s_{k_2}), (s_{k_2}, \ell_{k_2}, s_{k_3}), \dots, (s_{k_t}, \ell_{k_t}, s_{k_1}))$$

or equivalently

$$\mathcal{C} = s_{k_1} \xrightarrow{\ell_{k_1}} s_{k_2} \xrightarrow{\ell_{k_2}} \dots \xrightarrow{\ell_{k_{t-1}}} s_{k_t} \xrightarrow{\ell_{k_t}} s_{k_1}$$

where no two edges in the sequence are equal and for all $i, j \in \{1, \dots \ell\}$ with $i \neq j$, $s_{k_i} \neq s_{k_j}$.

Definition 3.5. A *cycle* is an equivalence class of simple closed paths where the equivalence relation $\sim_P$ is defined as follows: we have $\mathcal{C}_1 \sim_P \mathcal{C}_2$ provided that we can cyclically permute $\mathcal{C}_1$ to be $\mathcal{C}_2$.

For a shortlex geodesic ray $c : [0, \infty) \to \Gamma$, let $\mathcal{S}(c(n))$ denote the state of $c(n)$ in W. Furthermore, let $E_c(n) := ((\mathcal{S}(c(n)), f_c(n + 1), \mathcal{S}(c(n + 1))))$.

Definition 3.6. Let C_i be a cycle in W. We say that a shortlex geodesic ray $c : [0, \infty) \to \Gamma$ *terminates* in a cycle C_i of W provided there exists N such that for all $n > N$, $E_c(n)$ is contained in C_i. The minimum such N is called the *terminating value.*

Lemma 3.7. *Let $C = ((s_{k_1}, \ell_{k_1}, s_{k_2}), (s_{k_2}, \ell_{k_2}, s_{k_3}), \dots, (s_{k_t}, \ell_{k_t}, s_{k_1}))$ be a cycle in W such that the shortlex geodesic ray $c : [0, \infty) \to \Gamma$ terminates in C with terminating value N and $E_c(N + 1) = (s_{k_1}, \ell_{k_1}, s_{k_2})$. Then, it follows that for each $1 \leq j \leq t$, $E_c(N + j) = (s_{k_j}, \ell_{k_j}, s_{k_{j+1}})$.*

Proof. We have that the edge $E_c(N + 2)$ is going out of s_{k_2} and is contained in C. By our definition, a simply closed path cannot contain more than one edge going out of a state, so it follows that $E_c(N + 2) = (s_{k_2}, \ell_{k_2}, s_{k_3})$. The same reasoning can be applied for $E_c(N + 3), E_c(N + 4), \dots, E_c(N + t)$, giving us the result. ■

Lemma 3.8. *If two shortlex geodesic rays $c_1, c_2 : [0, \infty) \to \Gamma$ terminate in the same cycle, then they must represent the same end behavior.*

Proof. Suppose that c_1, c_2 both terminate in the cycle

$$C = ((s_{i_1}, \ell_{i_1}, s_{i_2}), \ldots, (s_{i_{k-1}}, \ell_{i_{k-1}}, s_{i_k}), (s_{i_k}, \ell_{i_k}, s_{i_1})).$$

Let M_1 be the minimum such that for all $n \geq M_1$, $E_{c_1}(n)$ is contained in C. Define M_2 in the same way for c_2. Then, we have that $E_{c_1}(M_1) = (s_{i_\ell}, \ell_{i_\ell}, s_{i_\ell})$ and $E_{c_2}(M_2) = (s_{i_m}, \ell_{i_m}, s_{i_m})$. Without loss of generality, suppose that $\ell \leq m$. Then, we can apply Lemma 3.7 to get that $E_{c_1}(M_1 + m - \ell) = E_{c_2}(M_2)$, which means that for all $t > 0$, $c_1(M_1 + m - \ell + t) = c_2(M_2 + t)$. Thus, c_1, c_2 represent the same end behavior. ■

Lemma 3.9. *No two cycles in W share a state if and only if the total number of different end-behaviors in Γ is finite.*

Proof. Let $C_1, \ldots, C_k$ be the cycles in W. Suppose that there are no cycles in W that share a state. Our goal is to show that every infinitely long shortlex geodesic word terminates in some C_i.

Let $w : [0, \infty) \to \Gamma$ be a shortlex geodesic. Since W has finitely many states, we can take a large enough prefix of w such that a state is repeated, which means we've gone around a cycle, which we can call C_1. Now, suppose that a large enough prefix escapes C_1. Eventually, w will have to repeat a state, so it will have traversed through a cycle C_2. Now, we note that w cannot reenter C_1 because in doing so, it would have to traverse another cycle which would share a state with C_1. If w leaves C_2, it will have to enter a new cycle C_3. This process can only happen finitely many times because there are finitely many cycles and w cannot reenter a cycle it has exited. Thus, w must eventually terminate in a cycle C_i.

Now, suppose that there exist cycles C_1, C_2 such that

$$C_1 = s \xrightarrow{\ell_1} s_1 \xrightarrow{\ell_2} \ldots \xrightarrow{\ell_{k-1}} s_k \xrightarrow{\ell_k} s$$

and

$$C_2 = s \xrightarrow{t_1} m_1 \xrightarrow{t_2} \ldots \xrightarrow{t_{j-1}} m_j \xrightarrow{t_j} s$$

with $t_1 \neq \ell_1$. Suppose that w is a sequence of labels (or equivalently a word) from the start state to the state s. For each $n \in \mathbb{N}$, let $w_n = (\ell_1 \ell_2 \ldots \ell_k)^n (t_1 t_2 \ldots t_j)^n$.

We can consider the infinitely long shortlex geodesic word ww_n^∞ (where w_n^∞ means w_n repeating forever). For each n, ww_n^∞ represents a different end behavior, so there must be infinitely many end behaviors. ■

Now, we can define a group action of Γ on $\partial\Gamma$ by left multiplication on the infinitely long word in $\partial\Gamma$, which we then reduce. We can now relate end behaviors to orbits under this group action.

Lemma 3.10. *If two geodesic rays $c_1, c_2 : [0, \infty) \to \Gamma$ represent the same end behavior then they are part of the same orbit under the group action defined above.*

Proof. Suppose there exist T_1, T_2 such that for all $t \in \mathbb{Z}$, $f_{c_1}(T_1 + t) = f_{c_2}(T_2 + t)$. Then, we can left-multiply c_1 by $c_2(T_2)c_1^{-1}(T_1)$ to get the following infinitely long word:

$$f_{c_2}(1) \ldots f_{c_2}(T_2) f_{c_1}(T_1 + 1) f_{c_1}(T_1 + 2) \ldots$$

which is equal to c_2. Thus, c_1 and c_2 are in the same orbit. ■

Lemma 3.11. *G is finite if and only if W contains no cycles.*

Proof. Suppose that W contains a cycle $((s_{k_1}, \ell_{k_1}, s_{k_2}), (s_{k_2}, \ell_{k_2}, s_{k_3}), \ldots, (s_{k_t}, \ell_{k_t}, s_{k_1}))$. Without loss of generality, suppose that s_{k_1} is the start state. Furthermore, suppose that there's a path from s_{k_1} to an accept state with edges labeled $b_1, b_2, \ldots, b_m$. Then, for each $n \in \mathbb{Z}$, the word $(\ell_{k_1}\ell_{k_2}\ldots\ell_{k_t})^n b_1 b_2 \ldots b_m$ must be a shortlex geodesic word accepted by W. Since each shortlex geodesic word uniquely represents an element of G, it follows that G must be infinite.

Now, suppose that G is infinite. Then, we can find shortlex geodesic words of arbitrary length. In particular, we can find one whose length is larger than the number of states in W. Thus, in order for W to accept such a word, it would have to contain a path longer than the total number of states, so it must contain a cycle. ■

Now we're ready to bring together these results to prove Theorem 1.1. We restate the result here for convenience.

Theorem 1.1. *$\partial\Gamma$ is finite if and only if no distinct cycles in W share a state.*

Proof. For the forward direction, we can split it into two cases: when $\partial\Gamma$ is empty and when it's non-empty. If $\partial\Gamma$ is empty, then it follows that G must be finite. Thus, it follows from Lemma 3.11 that W doesn't contain any cycles, giving us the result. Now, if $|\partial\Gamma|$ is finite and non-empty, it follows from Lemma 3.3 that the total number of end behaviors is finite. Using Lemma 3.9, this implies that no two cycles in W share a state.

For the reverse direction, we can consider the case where W has no cycles separately. In this case, it follows that G must be finite, which means that $\partial\Gamma$ is empty. Now, if W contains cycles and no two cycles share a state, then by Lemma 3.9, the number of end behaviors is finite. This means that by Lemma 3.10 the number of orbits given by Γ acting on its boundary must be finite. Suppose, for contradiction, that the boundary contains infinitely many points. It follows that the boundary must be uncountable due to Lemma 2.8. However, the boundary is covered by the union of all orbits, so since there are a finite number of orbits and every orbit is countable, we have a contradiction. Thus, the number of boundary points must be infinite. ■

Recall that as a consequence, we have the following:

Corollary 1.2. *G is virtually cyclic if and only if no distinct cycles in W share a state.*

Proof. By Theorem 1.1, it suffices to show that G being virtually cyclic is equivalent to $\partial\Gamma$ being finite. This follows from Theorem 2.28 of [5]. ■

4. Concluding Remarks and Future Work

In this paper, we've only tried to characterize the cardinality of the boundary using the automatic structure. It turns out that we can endow the Gromov boundary with a topology[6], so one can ask whether there is any way of characterizing connectivity of the boundary using the automatic structure. In particular, the goal here is to find some sort of computable procedure that takes in as input the word acceptor and multiplier automata and decides if the boundary of the corresponding group is connected.

In addition to the results described in the previous section, we have spent time trying to tackle this problem of characterizing connectedness. Our main approach has been to try to

[6]See [1, III.H.3]

look at the *ends*[7] of the Cayley graph since the number of ends corresponds exactly with the number of connected components in the boundary. Thus, the question becomes whether you can use the automatic structure to determine if for any two geodesic rays $c_1, c_2 : [0, \infty) \to \Gamma$ based at the identity and for every R, we can find a path from $c_1(R+k)$ to $c_2(R+k)$ that avoids the ball of radius R about the identity, where k is some constant that depends on the Cayley graph.

In the future, the we hope to continue thinking about this question and possibly find some sort of algorithm to decide whether the boundary is connected.

References

[1] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[2] Matt Clay and Dan Margalit, editors. *Office Hours with a Geometric Group Theorist*. Princeton University Press, 2017.

[3] C. Druţu and M. Kapovich. *Geometric Group Theory*. Colloquium Publications. American Mathematical Society, 2018.

[4] David B. A. Epstein, James W. Cannon, Derek F. Holt, Silvio V. F. Levy, Michael S. Paterson, and William P. Thurston. *Word processing in groups*. Jones and Bartlett Publishers, Boston, MA, 1992.

[5] Ilya Kapovich and Nadia Benakli. Boundaries of hyperbolic groups. In *Combinatorial and geometric group theory (New York, 2000/Hoboken, NJ, 2001)*, volume 296 of *Contemp. Math.*, pages 39–93. Amer. Math. Soc., Providence, RI, 2002.

Mathematics Department, University of Michigan, Ann Arbor, MI 48109
Email address: shukulab@umich.edu

[7] See [1, I.8]