INHOMOGENEOUS DIOPHANTINE APPROXIMATION: WINNING PROPERTY AND BEYOND

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Abstract. We extend the winning property of weighted badly approximable vectors with weights $(i, ..., i, l), i \geq l$ in $\mathbb{R}^n$ from homogeneous to simultaneous inhomogeneous. In addition, we show the winning property of these vectors on certain types of hyperplanes in $\mathbb{R}^n$. Besides, we extend the winning property of weighted badly approximable vectors with any weights on non-degenerate curves from homogeneous to dual inhomogeneous.

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1. Introduction

1.1. Dirichlet's theorem and badly approximable numbers. .

It is well-known that $\mathbb{Q}$ is dense in $\mathbb{R}$. But how well can a real number be approximated by rational numbers? Dirichlet somehow provided an answer to this by proving the following theorem

Theorem 1.1 (Dirichlet)**.** *When $c = 1$, $\forall x \notin \mathbb{Q}, \exists$ infinitely many reduced fractions $\frac{p}{q}$ such that $|x - \frac{p}{q}| < \frac{c}{q^2}$*

This theorem says that any irrational number can be approximated at a rate of 1 over the square of the denominators. Another question then comes up naturally: can this approximation be improved?.

On one hand, it can be checked that the numbers that can be approximated by the rate of $\frac{1}{q^p}$ for some $p > 2$ is of measure zero. In other words, the index '2' in the approximation cannot be improved.

On the other hand, fortunately, we can improve the constant c in Theorem 1.1 to arbitrarily small for almost every real number. We call the numbers avoiding this improvement badly approximable numbers, defined as follows

Definition 1.1. *$x \in \mathbb{R}$ is badly approximable if*

$$c(x) := \inf_{q \in \mathbb{N} \setminus \{0\}} q\|qx\| > 0,$$

where $\|qx\| := \inf_{p \in \mathbb{Z}} |qx - p|$.

We usually denote the set of badly approximable numbers as **Bad**. The set of badly approximable numbers is non-empty, e.g. $c(\frac{\sqrt{5}+1}{2}) = \frac{1}{\sqrt{5}} > 0$ and the golden ratio is badly approximable.

Though it is proven by Khintchine that the Lebesgue measure of **Bad** is 0, the structure inside **Bad** is very interesting.

1.2. Hausdorff dimension. .

Continuing from last subsection, **Bad** is "thick" over $\mathbb{R}$. In particular, **Bad** is of full Hausdorff dimension. We can show this thickness by constructing a $(R, 2)$-Cantor set inside **Bad** for any sufficiently large integer R. If so, since it is known that the Hausdorff dimension of a $(R, 2)$-Cantor set is $\frac{\ln(R-2)}{\ln R}$, the Hausdorff dimension of **Bad** is no less than 1 when $R \to \infty$, while the other direction of the inequality holds trivially.

Before giving the details of the construction, we would like to give some other definitions. How do we make sure that a real number is badly approximable? The property of irrational numbers is too vague for us to explicitly compute $c(x), x \notin \mathbb{Q}$. Therefore, we'd rather start from rational numbers and remove some dangerous intervals.

Definition 1.2. *Given $\delta > 0, \frac{p}{q} \in \mathbb{Q}$, the dangerous interval is*

$$\Delta_\delta(\frac{p}{q}) := \{x \in \mathbb{R} : |x - \frac{p}{q}| < \frac{\delta}{q^2}\}.$$

It can be checked that after avoiding all the dangerous intervals for certain δ, the number must be badly approximable, i.e.

$$\mathbb{R} \setminus \cup_{\frac{p}{q} \in \mathbb{Q}} \Delta_\delta(\frac{p}{q}) \subset \mathbf{Bad}.$$

Then, the goal of the Cantor set construction will be removing these dangerous intervals layer by layer. The key idea of the construction is the following lemma

Lemma 1.1 (Simplex Lemma)**.** *For any integer $R \geq 4$, and an interval I_n of length R^{-n+1}, there is* ***at most one*** *$p/q \in \mathbb{Q}$ such that $R^{(n-3)/2} \leq q < R^{(n-2)/2}$ and $I_n \cap \Delta_{1/2}(p/q) \neq \emptyset$.*

We can do so because the rational numbers are relatively sparse when the denominator no larger than a certain bound, which is unique for rational numbers. Moreover, since R is sufficiently large, that unique dangerous interval intersects at most 2 of the subintervals of I_n. Thus, the $(R, 2)$-Cantor set follows from this construction.

1.3. **Schmidt's game and winning property.** .

In the 1960s, Schmidt provided us a even stronger property than full Hausdorff dimension, called winning property. It is defined via the notion of Schmidt games.

Schmidt games are defined as follows. There are two players A and B, together with two indexes $0 < \alpha, \beta < 1$. At first the player B would choose a ball B_0 in a metric space M. Then inductively, in n-th round, the player A would choose a ball $A_n \subset B_{b-1}$ of radius $\alpha|B_{n-1}|$, while the player B also chooses a ball $B_n \subset A_n$ of radius $\beta|A_n|$. Thus, these balls would finally converge to a limit point $x_0 = \cap_n A_n$.

Definition 1.3. *A subset $X \subset M$ is called $(\alpha, \beta)-$winning if the player A can always make sure $x_0 \in X$ regardless of how the player B plays.*

In addition, if for a fixed α, X is $(\alpha, \beta) -$ winning for all $\beta \in (0, 1)$, X is called $\alpha -$ winning, or in short, winning.

As mentioned before, this winning property is stronger than full Hausdorff dimension. It is also closely related to Diophantine approximation. In fact, the classical examples of winning sets in Euclidean spaces are the set of badly approximable vectors, which is the higher dimensional analogue of badly approximable numbers as follows.

Definition 1.4. *Given $\boldsymbol{r} \in \mathbb{R}^n$ such that $0 < r_i \leq 1$ and $\sum_{i=1}^n r_i = 1$, we define*

$$\boldsymbol{Bad}(\boldsymbol{r}) := \{\boldsymbol{x} \in \mathbb{R}^n : \inf_{q \in \mathbb{N}\setminus\{0\}} q \max_{1 \leq i \leq n} \|qx_i\|^{1/r_i} > 0\}.$$

Note that the definition involves a weight vecctor $\mathbf{r}$, which gives the distribution of the 'badness' along different directions. It is shown in [10] that the unweighted bad, i.e. $\mathbf{Bad}(1/n, 1/n, ..., 1/n)$ is winning. However, it seems quite difficult to step from unweighted to weighted. It is not until recently[5] that the set of weighted badly approximable vectors is proven to be winning.

It is also worth noticing that besides $\mathbb{R}^n$, the winning property of weighted bad is also proven in various situations, including non-degenerate curves[6] and certain hyperplanes[2], which will be introduced along the demonstration of our work.

1.4. **Inhomogeneous Diophantine Approximation.** .

There are two equivalent definitions of weighted bad. The one we have introduced is also called simultaneous badly approximable vector. The other one, or dual badly approximable vectors. is defined via the following lemma:

Lemma 1.2. *Given $\boldsymbol{r} \in \mathbb{R}^n$ such that $0 < r_i \leq 1$ and $\sum_{i=1}^n r_i = 1$,$\boldsymbol{x} \in \boldsymbol{Bad}(\boldsymbol{r})$ if and only if*

$$\inf_{a_0 \in \mathbb{Z}, \boldsymbol{a} \in \mathbb{Z}^n \setminus\{0\}} |a_0 + \boldsymbol{a} \cdot \boldsymbol{x}| \max_{1 \leq i \leq n} |a_i|^{1/r_i} > 0.$$

Correspondingly, if we are first giving the definition of the integer norm.

Definition 1.5. *If $y \in \mathbb{R}$*

$$|y|_{\mathbb{Z}} := \inf_{p \in \mathbb{Z}} |y + p|. \tag{1.1}$$

There are two distinct definitions of inhomogeneous badly approximable vectors:

Definition 1.6 (Simultaneous Inhomogeneous Bad). *Given $\Theta = (\theta_i)$, we define*

$$\boldsymbol{Bad}_\Theta(\boldsymbol{r}) := \{\boldsymbol{x} \in \mathbb{R}^n : \inf_{q \in \mathbb{Z} \setminus \{0\}} \max_{1 \le i \le n} |qx_i - \theta_i|_{\mathbb{Z}}^{1/r_i} |q| > 0\}.$$

Definition 1.7 (Dual Inhomogeneous Bad). *Given $\theta \in \mathbb{R}$,*

$$\boldsymbol{Bad}_\theta(\boldsymbol{r}) := \{\mathbf{x} \in \mathbb{R}^n : \inf_{\boldsymbol{a}=(a_i) \in \mathbb{Z}^n \setminus \{0\}} |\boldsymbol{a} \cdot \boldsymbol{x} + \theta|_{\mathbb{Z}} \max_{1 \le i \le n} |a_i|^{1/r_i} > 0\}.$$

The primary goal of this project is to extend the recent results from homogeneous to inhomogeneous.

2. Main Theorems

2.1. Simultaneous Inhomogeneous Bad. .

In the last decade, many attempts and efforts have been made to prove the winning properties of the set of weighted bad vectors. One of the breakthroughs is to apply the method of attaching a hyperplane to each rational vector and further classifying rational numbers[1], so that we can still control the width of the strip to remove and adapt the proof in [10]. For now, the best result it accomplishes is to allow at most two distinct indexes in the weights, or explicitly the weight of the form $\mathbf{r} \in \mathbb{R}^n$ such that $r_1 = ... = r_{n-1} \ge r_n$[8, Theorem 1.5]. In this paper, we extend this winning property to simultaneous inhomogeneous bad using the technique from [2]. Formally, we get the following result, which extends [8, Theorem 1.5] from homogeneous to inhomogeneous, i.e., from $\Theta = 0$ to any $\Theta \in \mathbb{R}^d$.

Theorem 2.1. *Given $\Theta = (\theta_i), \boldsymbol{r} \in \mathbb{R}^d$ such that $r_1 = ... = r_{d-1} \ge r_d$, we have $\boldsymbol{Bad}_\Theta(\boldsymbol{r})$ is hyperplane absolute winning.*

Notice that hyperplane absolute winning is a property even stronger than winning, which will be introduced later.

In [2, Theorem 1.2], the winning property of inhomogeneous bad with such weights is also proven within certain lines in $\mathbb{R}^2$. We extend this result to higher dimensions by applying the method in [8].

Theorem 2.2. *Suppose $\Theta = (\theta_i) \in \mathbb{R}^d$ and the hyperplane*

$$L_{\boldsymbol{e},b} := \{(x_1, ..., x_d) : x_d = \sum_{k=1}^{d-1} e_k x_k + b\}.$$

If $i \ge l > 0, (d-1)i + l = 1$ and

$$\exists\ \epsilon > 0,\ \ s.t.\ \ c_0 := \lim_{q \to \infty} q^{\frac{1}{l} - \epsilon} \max\{|qb|_{\mathbb{Z}}, |qe_k|_{\mathbb{Z}}^{k=1,...,d-1}\} > 0,$$

then $\boldsymbol{Bad}_\Theta(i, ..., i, l)$ is hyperplane absolute winning within $L_{\boldsymbol{e},b}$.

2.2. **Dual Inhomogeneous Bad.** . The dual inhomogeneous bad sets are equally interesting to study, and much less is known about them. A recent breakthrough [6] was to show weighted bad sets to be absolute winning inside *nondegenerate* curves. Also, in [5], weighted bad vectors in $\mathbb{R}^n$ was shown to be hyperplane winning. The methods in both of these papers use homogeneous dynamics, to be specific quantitative nondivergence estimates from the famous work [9].

In this paper, we prove the dual inhomogeneous analogue of the result in [6, Theorem 1.1] as follows.

Theorem 2.3. *Let $\theta \in \mathbb{R}$ be the dual inhomogeneous index, $\mathbf{r}$ be n tuple of weights, and $\varphi : U \subset \mathbb{R} \to \mathbb{R}^n$, be a map defined on an open interval. Suppose φ be an analytic map that is nondegenerate. Then $\varphi^{-1}(\boldsymbol{Bad}_\theta(r))$ is (hyperplane) absolute winning within U.*

Note that hyperplane absolute winning, though weaker in general, is equivalent to absolute winning in one-dimensional case.

3. Proof of Theorem 2.1

3.1. **Preliminaries.** .

Hyperplane absolute winning (HAW) is introduced in [7] and it is stronger than winning. The definition can be checked in [8]. To prove HAW, we need another winning property called hyperplane potential winning(HPW). HPW is defined via the following hyperplane potential game.

There are two players A and B. Given two parameters, $\beta \in (0,1)$ and $\gamma > 0$. In the i-th round, the player B chooses a ball B_i of radius ρ_i such that $\rho_i \geq \beta\rho_{i-1}$ and $B_i \subset B_{i-1}$(if $i = 0$, B_0 can be any ball). Meanwhile, the player A chooses a countable family of hyperplane neighborhood $\{\mathcal{H}_{i,k}^{\delta_{i,k}}\}_{k\in\mathbb{N}}$ such that

$$\sum_{k\in\mathbb{N}} \delta_{i,k} \leq (\beta\rho_i)^\gamma. \tag{3.1}$$

Here the hyperplane neighborhood is defined as follows.

Definition 3.1. *Given a hyperplane $\mathcal{H}$ and $\delta > 0$, the hyperplane neighborhood*

$$\mathcal{H}^\delta := \{x : d(x, H) \leq \delta\}.$$

Now we can define the corresponding winning property.

Definition 3.2. *A set S is called $(\beta, \gamma)-$winning if the player A can always make sure*

$$\bigcap_{i\in\mathbb{N}} B_i \cap (S \cup \bigcup_{i,k\in\mathbb{N}} \mathcal{H}_{i,k}^{\delta_{i,k}}) \neq \emptyset, \tag{3.2}$$

no matter how the player B plays.

In addition, S is HPW if it is $(\beta,\gamma)-$winning for any $\beta \in (0,1), \gamma > 0$.

The following proposition as stated in [8, Proposition 2.2] can be used to show HAW via HPW.

Proposition 3.1 ([8]). *$S \subset \mathbb{R}^d$ is HAW if and only if HPW.*

3.2. **Notations and Results from** [8]. .

Fix $R > 0$ large enough.

For each $P \in \mathbb{Q}^d$, we attach a hyperplane

$$\mathcal{H}_P := \{x \in \mathbb{R}^d : F_P(x) = \sum_{m=1}^{d} a_m x_m + C = 0\}, \tag{3.3}$$

while $a_m, C \in \mathbb{Z}$ satisfies the following property:

$$a_m p_m \in q\mathbb{Z}, \forall m,$$

$$|a_m| \leq q^{r_m}, 1 \leq m \leq d,$$

$$F_P(P) = 0,$$

which can be done according to [8, Section 3.1]. And a_m, C will be related to $P \in \mathbb{Q}^d$ by default.

Let

$$s := \max_{1 \leq i \leq d} r_i, \tag{3.4}$$

$$c := \frac{1}{8} d^{-2} \rho_0 R^{-18d^2}, \tag{3.5}$$

$$\Delta(P) := \{x \in \mathbb{R}^d : |x_m - \frac{p_m}{q}| < \frac{c}{q^{1+r_m}}, \forall 1 \leq m \leq d\}, \tag{3.6}$$

$$H(P) := q \max_{1 \leq i \leq d} |a_i|,$$

$$H_n = dc\rho_0^{-1} R^n,$$

$$\mathcal{P}_n := \{P \in \mathbb{Q}^d, H_n \leq H(P) < H_{n+1}\},$$

$$O_{n,k} := H_n^{1/(1+s)} R^{d(k-2)+12d^2},$$

$$\mathcal{P}_{n,1} := \{P \in \mathcal{P}_n : H_n^{1/(1+s)} \leq q < H_n^{1/(1+s)} R^{12d^2}\},$$

$$\mathcal{C} := \{P \in \mathcal{P} : \nexists P' \in \mathcal{P}, \textbf{s.t.} \Delta(P) \subset \Delta(P')\},$$

$$\mathcal{P}_{n,k} := \{P \in \mathcal{P}_n : O_{n,k} \leq q < O_{n,k+1}\}, k \geq 2,$$

$$B \subset \mathbb{R}^d, \mathcal{C}_{n,k}(B) := \{P \in \mathcal{P}_{n,k} \cap \mathcal{C}, \Delta(P) \cap B \neq \emptyset\}.$$

Without loss of generality, we want a larger R than that in [8] such that

$$(R^\gamma - 1)^{-1} < \frac{1}{2} (\frac{\beta^2}{2})^\gamma. \tag{3.7}$$

Also, we list the key proposition, [8, Corollary 4.6], to remove all these dangerous cubes.

Proposition 3.2. *$\exists$a hyperplane $E_k(B) \subset R^d$ such that $\forall P \in \mathcal{C}_{n+k,k}(B), \Delta(P) \cap B \subset E_k(B)^{\rho_0 R^{-n-k}}$.*

3.3. From Homogeneous to Inhomogeneous. .

We introduce some new notations: $c' = \frac{1}{6}cR^{-2}$ and $H'_n = 3c'\rho_0^{-1}R^n$.Let $\Theta = (\gamma_1, ..., \gamma_d)$,

$$\mathcal{V}_n = \{P \in \mathbb{Q}^d : H'_n \leq q^{1+i} < H'_{n+1}\}$$

and

$$\Delta_\Theta(P) = \{x \in \mathbb{R}^d : |x_i - \frac{p_i + \gamma_i}{q}| < \frac{c'}{q^{1+r_i}}\}.$$

Lemma 3.1. *If $diam(B) \leq \rho_0 R^{-n+1}$ and $\forall k < n, \forall P \in \mathcal{P}_k, B \cap \Delta(P) = \emptyset$, then there is at most one $v \in \mathcal{V}_n$ such that $\Delta_\Theta(v) \cap B \neq \emptyset$. Moreover, the corresponding $\Delta_\Theta(v)$ can be contained in a hyperplane neighborhood $G(B)^{\frac{1}{3}\rho_0 R^{-n}}$.*

Proof. Given such B, and suppose there are two $v_1, v_2 \in \mathcal{V}_n$ such that $\Delta(v_s) \cap B \neq \emptyset, s = 1, 2$. We let $x_s \in \Delta_\Theta(v_s) \cap B$ and WLOG, $q_1 \geq q_2$

Then by assumption we have

$$|q_s x_{i,s} - (p_{i,s} + \gamma_i)| \leq \frac{c'}{q_s^{r_i}}, s = 1, 2, i = 1, 2, ..., d.$$

Thus,

$$\begin{aligned} |(q_1 - q_2)x_{i,1} - (p_{i,1} - p_{i,2})| &\leq q_2|x_{i,1} - x_{i,2}| + |q_1 x_{i,1} - p_{i,1} - \gamma_i| + |q_2 x_{i,2} - p_{i,2} - \gamma_i| \\ &\leq q_2 \rho_0 R^{-n+1} + \frac{2c'}{q^{r_i}}. \end{aligned} \tag{3.8}$$

First, we show that $q_1 = q_2$. Suppose not then

$$\begin{aligned} |x_{i,1} - \frac{p_{i,1} - p_{i,2}}{q_1 - q_2}| &\leq \frac{1}{q_1 - q_2}(q_2 \rho_0 R^{-n+1} + \frac{2c'}{q_2^{r_i}}) \\ &\leq \frac{1}{(q_1 - q_2)^{1+r_i}}(q_1^{r_i} q_2 \rho_0 R^{-n+1} + 2c'R) \\ &\leq \frac{1}{(q_1 - q_2)^{1+r_i}}(\frac{c}{2} + \frac{1}{3}cR^{-1}) \\ &\leq \frac{c}{(q_1 - q_2)^{1+r_i}}. \end{aligned} \tag{3.9}$$

Thus, let $P_0 = (\frac{p_{i,1} - p_{i,2}}{q_1 - q_2})_{i=1,...,d}$, then $x_1 \in \Delta(P_0)$. Now denote n_0 such that $P_0 \in \mathcal{P}_{n_0}$ and by assumption $n_0 \geq n$, otherwise $\Delta(P_0) \cap B \neq \emptyset$ violates the assumption stated in the theorem. Now

$$(q_1 - q_2)^{1+i} \geq H_{n_0} \geq H_n = dc\rho_0^{-1}R^n.$$

However, it contradicts the fact that

$$(q_1 - q_2)^{1+i} \leq q_1^{1+i} \leq H'_{n+1} = \frac{1}{2}c\rho_0^{-1}R^{n-1}.$$

Therefore, $q_1 = q_2$. Moreover, by (3.8) we have that

$$|p_{i,1} - p_{i,2}| \leq q_2 \rho_0 R^{-n+1} + \frac{2c'}{q^{r_i}} \leq c + 2c' < 3c < 1.$$

Thus, $p_{i,1} = p_{i,2}, i = 1, ..., d$ and $v_1 = v_2$.

The second part of the theorem is true since $\Delta_\Theta(v)$ is a box and contained in a hyperplane neighborhood $G(B)^{\frac{1}{3}\rho_0 R^{-n}}$. □

3.4. Proof of Theorem 2.1. .

Let $i(n)$ be the smallest number of round such that $\beta R^{-n}\rho_0 < |B_{i(n)}| \leq R^{-n}\rho_0$. In [8], the player A chooses $\{E_k(B_{i(n)})^{\rho_0 R^{-n-k}}\}_{k\in\mathbb{N}}$, where $E_k(B_{i(n)})$ is as described in Proposition 3.2.

Let $B'_{i(n)} = B_{i(n)} - \cup_{k=1}^n \cup_{P\in\mathcal{P}_k} \Delta(P)$, then By Lemma 3.1 $\forall n, \exists G_n, s.t. \cup_{v\in\mathcal{V}_{n+1}} \Delta_\theta(v) \cap B'_{i(n)} \in G_n^{\frac{1}{3}\rho_0 R^{-n-1}}$. Thus, we add this hyperplane neighborhood to the neighborhoods Alice chosen in the $i(n)$-th round. It is a legal move because

$$(\frac{1}{3}\rho_0 R^{-n-1})^\gamma + \sum_{k=1}^{\infty}(\rho_0 R^{-n-k})^\gamma < 2\sum_{k=1}^{\infty}(\rho_0 R^{-n-k})^\gamma < (\beta\rho_i)^\gamma.$$

Now

$$\begin{aligned}\cap_{i\in\mathbb{N}} B_i \subset\ & \mathbf{Bad}(r) \cap \mathbf{Bad}_\Theta(r) \cup (\cup_{n=1}^\infty (\cup_{v\in\mathcal{V}_{n+1}} \Delta_\Theta(v) \cap B'_{i(n)}) \\ & \cup (\cup_{k=1}^\infty \cup_{P\in\mathcal{C}_{n+k,k}} \Delta(P) \cap B_{i(n)})) \\ \subset\ & \mathbf{Bad}(r) \cap \mathbf{Bad}_\Theta(r) \cup (\cup_{n=1}^\infty G_n^{\frac{1}{3}\rho_0 R^{-n}} \cup \cup_{k=1}^\infty E_k(B_{i(n)})^{\rho_0 R^{-n-k}}).\end{aligned} \tag{3.10}$$

Thus, $\mathbf{Bad}(r) \cap \mathbf{Bad}_\Theta(r)$ is HPW and so $\mathbf{Bad}_\Theta(r)$ containing this set is also HPW and by Proposition 3.1 HAW. □

4. Proof of Theorem 2.2

4.1. Notation. .

For $n \in \mathbb{N}, x \in \mathbb{R}^n$, we will denote by $x = (x_1, \cdots, x_n)$.

For $P \in \mathbb{Q}^d$, we will denote it by $P = (\frac{p_1}{q}, \cdots, \frac{p_d}{q})$.

Let i, l, e_m, b, c_0 and $,\epsilon$ be the ones appearing in Theorem 2.2.

4.2. Some definitions. .

For each $P \in \mathbb{Q}^d$, we attach a hyperplane

$$\mathcal{H}_P := \{x \in \mathbb{R}^d : F_P(x) = \sum_{m=1}^{d} a_m x_m + C = 0\}, \tag{4.1}$$

while $a_m, C \in \mathbb{Z}$ satisfies the following property:

$$a_m p_m \in q\mathbb{Z}, \forall m,$$

$$|a_m| \leq q^i, 1 \leq m \leq d-1,$$

$$|a_d| \leq q^l,$$

$$F_P(P) = 0,$$

which can be done according to [8, Section 3.1]. And a_m, C will be related to $P \in \mathbb{Q}^d$ by default.

Let

$$\sigma := 1 + \sum_{k=1}^{d-1} |e_k|, c_1 := \min\{c_0, \frac{c_0}{(d-1)\max|x|_{max} + \rho_0}\},$$

where $|x|_{max} = \max\{\max|x_k|_{k=1,...,d-1}, x =\in B_0\}$.

Then choose

$$\lambda := \max(d, \frac{1}{l\epsilon}).$$

Meanwhile, choose $\mu > 0$ such that $R^\mu > \max(c_1^{-1}, d^2)\sigma$. Now let

$$c := \min(\frac{c_1}{12\sigma} d^{-2}\rho_0 R^{-18d\lambda}, (3\sigma d^{4-1/d}\rho_0^{1/d} R^{-2-12d})^{-\frac{d}{d-1}}), \tag{4.2}$$

$$\Delta(P) := \{x \in \mathbb{R}^{d-1} : |x_m - \frac{p_m}{q}| < \frac{c}{q^{1+i}}, \forall 1 \le m \le d-1\}, \tag{4.3}$$

$$H(P) := q \max_{1\le k\le d-1}\{|a_d e_k + a_k|\},$$

$$H_n = \sigma d c \rho_0^{-1} R^n,$$

$$\mathcal{P} := \{P \in \mathbb{Q}^d : B_0 \cap \Delta(P) \neq \emptyset, |b + \frac{\sum_{k=1}^{d-1} e_k p_k - p_d}{q}| < \frac{\sigma c}{q^{1+l}}\},$$

$$\mathcal{P}_n := \{P \in \mathcal{P}, H_n \le H < H_{n+1}\},$$

$$O_{n,k} := (\frac{H_n}{\sigma})^{1/(1+i)} R^{\lambda(k-2)+12d\lambda+\mu},$$

$$\mathcal{P}_{n,1} := \{P \in \mathcal{P}_n : (\frac{H_n}{\sigma})^{1/(1+i)} \le q < (\frac{H_n}{\sigma})^{1/(1+i)} R^{12\lambda d+\mu}\},$$

$$\mathcal{C} := \{P \in \mathcal{P} : \nexists P' \in \mathcal{P}, \textbf{s.t.} \Delta(P) \subset \Delta(P')\},$$

$$\mathcal{P}_{n,k} := \{P \in \mathcal{P}_n : O_{n,k} \le q < O_{n,k+1}\}, k \ge 2,$$

$$B \subset \mathbb{R}^{d-1}, \mathcal{C}_{n,k}(B) := \{P \in \mathcal{P}_{n,k} \cap \mathcal{C}, \Delta(P) \cap B \neq \emptyset\}.$$

4.3. **Extensions of Results from [2].** .

The HAW property is invariant under diffeomorphisms[7], so it suffices to prove that $\pi(\mathbf{Bad}(i,...,i,l) \cap L_{\mathbf{e},b}) \subset \mathbb{R}^{d-1}$ is HAW, where $\pi : \mathbb{R}^d \to \mathbb{R}^{d-1}, \pi(x_1,...,x_d) = (x_1,...,x_{d-1})$ is the projection onto the first $d-1$ coordinates. For now, we denote this projection $\mathbf{Bad}^L(i,...,i,l)$. Similarly, we define its dual inhomogeneous analogue and denote it as $\mathbf{Bad}^L_\theta(i,...,i,l)$.

We extend some useful lemmas from [2] to suit Theorem 2.2.

The following lemma is the analogue of [2, Lemma 6.2].

Lemma 4.1. *As defined above, removing the dangerous intervals would exclude all the good points, i.e.*

$$\mathbb{R}^{d-1} \setminus \cup_{P\in\mathcal{P}} \Delta(P) \subset \boldsymbol{Bad}^L(i,...,i,l).$$

Proof. Suppose $x \notin \mathbf{Bad}^L(i, ..., i, l)$, then there exists $P \in \mathbb{Q}^d$ such that

$$\forall 1 \leq m \leq d-1, |x_m - \frac{p_m}{q}| < \frac{c}{q^{1+i}},$$

$$|b + \sum_{m=1}^{d-1} e_m x_m - \frac{p_d}{q}| < \frac{c}{q^{1+l}}.$$

Thus, $x \in \Delta(P)$ and moreover,

$$|b + \frac{\sum_{k=1}^{d-1} e_k p_k - p_d}{q}| < |b + \sum_{m=1}^{d-1} e_m x_m - \frac{p_d}{q}| + \sum_{m=1}^{d-1} |e_m||x_m - \frac{p_m}{q}| < \frac{\sigma c}{q^{1+l}},$$

so $P \in \mathcal{P}$. □

The following lemma is the analogue of [2, Lemma 6.2].

Lemma 4.2. $\forall P \in \mathcal{P}, q^{-l\epsilon} H(P) \geq c_1$.

Proof. By assumption in Theorem 2.2,

$$q^{1-l\epsilon} \max(q^{-1}H, |a_d b + C|) \geq |a_d|^{\frac{1}{l}-\epsilon} \max(q^{-1}H, |a_d b + C|) \geq c_0.$$

If $q^{-l\epsilon}H \geq c_0 \geq c_1$, then the conclusion holds. Otherwise, $q^{1-l\epsilon}|a_d b + C| \geq c_0$, then for $x = (x_1, ..., x_{d-1}) \in B_0 \cap \Delta(P)$, which is nonempty since $P \in \mathcal{P}$,

$$q^{-l\epsilon}(d-1)H|x|_{max} \geq q^{1-l\epsilon}|\sum_{j=1}^{d-1}(a_d e_k + a_k)x_k| \overset{(4.1)}{\geq} c_0 - q|F_P(x_1, ..., x_{d_1}, b + \sum_{j=1}^{d-1} e_k x_k)|$$

$$\overset{(4.3)}{\geq} c_0 - 2d\sigma c \overset{(4.2)}{\geq} (d-1)c_1|x|_{max},$$

which implies that $q^{-l\epsilon}H(P) \geq c_1$ □

The following proposition is the analogue of [2, Lemma 6.5]

Proposition 4.1.

$$\mathcal{P} = \cup_{n=1}^{\infty} \mathcal{P}_n, \mathcal{P}_n = \cup_{k=1}^{n-1} \mathcal{P}_{n,k}.$$

Proof. The first statement follows from $H_1 \leq c_1$ and Lemma 4.2.

To prove the second half, it suffices to show that$q < O_{n,n}$. Notice that by Proposition 2.1, $\sigma q^{1+i} < \sigma(c_1^{-1}H_{n+1})^{\frac{1+i}{l\epsilon}} = H_n R \sigma c_1^{-1}(c_1^{-1}H_1 R^n)^{\frac{1+i}{l\epsilon}-1} < H_n R^{(1+i)(\lambda n+\mu)}$, which implies the result we want. □

4.4. **Attaching a Line to Rational Vectors.** .

Similar to [8], we also attach lines to rational vectors, but with different constrainets

Lemma 4.3. *For each $P \in \mathcal{P}$, we can also attach a line $L_P := \{P + \lambda v, \lambda \in R\}$, where v satisfies the following properties:*

$$\exists b \in Z, z \in \mathbb{Z}^d, \textbf{\textit{s.t.}}\ v = bP + z,$$

$$|v_m| \leq (d-1)q^{-i} =: w_m(d-1)q^{-i}, 1 \leq m \leq d-1$$

$$|v_d - \sum_{m=1}^{d-1} e_m v_m| \leq (d-1)q^{-1-l-i}H =: (d-1)w_d q^{-l}.$$

Proof. Let $D_k = |e_m a_d + a_m|, m < d$ and $D_d = a_d$, and choose j such that $w_j D_j q^{-r_j} = \max\{w_m D_m q^{-r_m}\}$. Moreover, let $y_m = x_m, m < d$ and $y_d = x_d - \sum_{m=1}^{d-1} e_m x_m$, we define $\prod_j := \{x \in R^d : |y_m| \leq w_m q^{-r_m}, j \neq m, |\sum_{m=1}^{d} a_m x_m| < 1\}$.

If $j = d$, then the volume $V(\prod_j) = |a_d|^{-1} \prod_{m=1}^{d-1} q^{-i} \geq q^{-1}$.

If $j \neq d$, then the volume $V(\prod_j) \geq qH^{-1} w_d q^{-1+i} = q^{-1}$.

Thus $\prod_j \cap \Lambda_P \neq \emptyset$, and $|\sum_{m=1}^{d} a_m v_m| = 0$, thus $|y_j| \leq (d-1) w_j q^{-r_j}$. □

Lemma 4.4. *Let w_d be as defined in Lemma 4.3, then $w_d \leq \sigma R^{-\lambda k - 10\lambda d - \mu}$.*

Proof. $w_d = q^{-1-i} H \leq O_{n,k}^{-1-i} H_{n+1} = \sigma R^{1-(1+i)(\lambda(k-2)+12d\lambda+\mu)} < \sigma R^{-\lambda k - 10\lambda d - \mu}$. □

4.5. **Avoiding Homogenous Dangerous Cubes.** .

Proposition 4.2. *For $B \subset \mathbb{R}^{d-1}$ with $diam(B) \leq \rho_0 R^{-n}$,$P_1, P_2 \in \mathcal{C}_{n+k,k}(B)$, then*

$$|F_{P_2}(P_1)| \leq \begin{cases} 3\sigma d^2 R^{2+\mu+12d\lambda} q_1^{-1} c, k = 1 \\ 3\sigma d^2 R^{k+\lambda+1} q_1^{-1} c, k > 1 \end{cases} \tag{4.4}$$

Proof.

$$\begin{aligned} |F_{P_2}(P_1)| =& |\sum_{m=1}^{d-1} (e_m a_{d,2} + a_{m,2})(\frac{p_{m,1}}{q_1} - \frac{p_{m,2}}{q_2}) \\ &+ a_{d,2}(\frac{\sum_{m=1}^{d-1} e_m p_{m,2} - p_{d,2}}{q_2} - \frac{\sum_{m=1}^{d-1} e_m p_{m,1} - p_{d,1}}{q_1})| \\ \leq& \frac{2d\sigma c}{q_1} \max(\frac{q_2}{q_1}, \frac{q_1}{q_2}, 1) + \frac{2(d-1)\sigma R^{-n} \rho_0 H}{q_2} \\ \leq& \begin{cases} \sigma(2d^2 R^2 + 2d) R^{\mu+12d\lambda} q_1^{-1} c, k = 1 \\ \sigma(2d^2 R^{k+1} + 2d) R^{\lambda} q_1^{-1} c, k > 1 \end{cases} \\ \leq& \begin{cases} 3\sigma d^2 R^{2+\mu+12d\lambda} q_1^{-1} c, k = 1 \\ 3\sigma d^2 R^{k+\lambda+1} q_1^{-1} c, k > 1 \end{cases} \end{aligned}$$

□

Corollary 4.1. *$P_1, P_2 \in \mathcal{C}_{n+k,k}(B)$, then $P_1 \in \mathcal{H}_{P_2}$.*

Proof. When $k = 1$, $q_1 |F_{P_2}(P_1)| < 1$ and since $q_1 F_{P_2}(P_1) \in Z$, we have $F_{P_2}(P_1) = 0$.

When $k > 1$, let the lines attached to these points to be $v_{m,j} = b_j \frac{p_{m,j}}{q_j} + z_{m,j}, m = 1, ..., d$ and suppose the opposite, i.e. $F_{P_2}(P_1) \neq 0$. Then there are two cases:

First, if $\mathcal{L}_{P_1}$ is parallel to H_{P_2}, then $\mathbf{a}_{P_2} \cdot \mathbf{v}_{P_1} = 0$. Thus, $b_1 q_1^{-1} \mathbf{a}_{P_2} \cdot \mathbf{p}_1 \in Z$, so $q_1 v_{m,1} F_{P_2}(P_1) \in Z, m = 1, ..., d$. Therefore, $q_1 F_{P_2}(P_1) \sum_{m=1}^{d} |v_{m,1}| \geq 1$ since $v_{P_1} \neq 0$.

However, this contradicts to the fact that

$$\begin{aligned} q_1 F_{P_2}(P_1) \sum_{m=1}^{d} |v_{m,1}| &\le 3\sigma^2 d^4 R^{k+\lambda+1} c \max(q_1^{-i}, w_d) \\ &\le \max(3\sigma^3 n d^4 R^{k+\lambda+1-\lambda k-10\lambda d} c, 3\sigma^2 d^{4-1/d} \rho_0^{1/d} R^{-2-12d} c^{1-1/d}) \\ &< 1. \end{aligned}$$

Thus, it always falls into the other case that L_{P_1} intersects with $\mathcal{H}_{P_2}$. Let the intersection point be $P_0 \in Q^d$. We want to show that $P \in \mathcal{P}$ and $\Delta(P_1) \subset \Delta(P_0)$ to get a contradiction. First, the denominator of P_0 can be no larger than $q_1|\mathbf{a}_{P_2} \cdot \mathbf{v}_{P_1}|$, which satisfies

$$\begin{aligned} \frac{q_0}{q_1} &\le |\mathbf{a}_{P_2} \cdot \mathbf{v}_{P_1}| \\ &= |a_{d,2}(v_{d,1} - \sum_{m=1}^{d-1} e_m v_{m,1}) + \sum_{m=1}^{d-1}(a_{m,2} + a_{d,2}e_m)v_{m,1}| \\ &\le (d-1)(w_{d,1}(q_2/q_1)^l + \sum_{m=1}^{d-1}(q_2/q_1)^i w_{d,2}) \\ &< d^2 R^\lambda \max(w_{d,1}, w_{d,2}) < \sigma d^2 R^{\lambda-\lambda k-10\lambda d-\mu} \\ &< R^{-\lambda k-6\lambda d}. \end{aligned}$$

And in particular $\frac{q_0}{q_1} < 1/2$. Now for any $1 \le m \le d-1$,

$$\begin{aligned} q_0^{1+i}|\frac{p_{m,0}}{q_0} - \frac{p_{m,1}}{q_1}| &= q_0^{1+i}|\frac{F_{P_2}(P_1)}{\mathbf{a}_{P_2} \cdot \mathbf{v}_{P_1}} v_{m,1}| \\ &\le dq_1|F_{P_2}(P_1)||\mathbf{a}_{P_2} \cdot \mathbf{v}_{P_1}|^i \\ &< 3d^3 R^{1-5\lambda} c \\ &< \frac{c}{2\sigma}. \end{aligned}$$

Then,

$$\begin{aligned} q_0^{1+l}|\frac{p_{d,0}}{q_0} - \frac{p_{d,1}}{q_1}| &= q_0^{1+l}|\frac{F_{P_2}(P_1)}{\mathbf{a}_{P_2} \cdot \mathbf{v}_{P_1}} v_{d,1}| \\ &\le q_0^{1+l}|\frac{F_{P_2}(P_1)}{\mathbf{a}_{P_2} \cdot \mathbf{v}_{P_1}}(v_{d,1} - \sum_{m=1}^{d-1} e_m v_{m,1})| + \sum_{m=1}^{d-1} |e_m| q_0^{1+i}|\frac{F_{P_2}(P_1)}{\mathbf{a}_{P_2} \cdot \mathbf{v}_{P_1}} v_{m,1}| \\ &< \frac{c}{2}. \end{aligned}$$

And

$$q_0^{1+l}|b+\frac{\sum_{m=1}^{d-1}e_mp_{m,0}-p_{d,0}}{q_0}|\leq(q_0/q_1)^{1+l}q_1^{1+l}|b+\frac{\sum_{m=1}^{d-1}e_mp_{m,1}-p_{d,1}}{q_1}|+q_0^{1+l}|\frac{p_{d,0}}{q_0}-\frac{p_{d,1}}{q_1}|$$
$$+q_0^{1+l}\sum_{m=1}^{d-1}|e_m||\frac{p_{m,0}}{q_0}-\frac{p_{m,1}}{q_1}|$$
$$<\sigma c.$$

Therefore, $P_0\in\mathcal{P}$. Moreover, given $x\in\Delta(P_1)$, we have that for any $1\leq m\leq d-1$

$$q_0^{1+i}|x_m-\frac{p_{m,0}}{q_0}|\leq q_0^{1+i}|x_m-\frac{p_{m,1}}{q_1}|+q_0^{1+i}|\frac{p_{m,0}}{q_0}-\frac{p_{m,1}}{q_1}|$$
$$<c,$$

which implies that $x\in\Delta(P_0)$ and we get the contradiction we want. □

Corollary 4.2. *$\exists$a hyperplane $E_k(B)\subset R^{d-1}$ such that $\forall P\in\mathcal{C}_{n+k,k}(B),\Delta(P)\cap B\subset E_k(B)^{\rho_0R^{-n-k}}$.*

Proof. Let $\hat{P}$ be the rational number such that $\hat{q}=\min\{q:P=\frac{\mathbf{p}}{q}\in\mathcal{C}_{n+k,k}(B)\}$ and $E_k(B):=\{(x_1,...,x_{d-1}):F_{\hat{P}}(x_1,...,x_{d-1},b+\sum_{m=1}^{d-1}e_mx_m)=\hat{a}_db+C_{\hat{P}}+\sum_{m=1}^{d-1}(\hat{a}_m+\hat{a}_de_m)x_m=0\}$. Then for any $x\in\Delta(P)\cap B$ for some $P\in\mathcal{C}_{n+k,k}$, its distance to $E_k(B)$ is

$$\frac{F_{\hat{P}}(x_1,...,x_{d-1},b+\sum_{m=1}^{d-1}e_mx_m)}{\sqrt{\sum_{m=1}^{d-1}|\hat{a}_m+e_m\hat{a}_d|^2}}\leq\frac{|\hat{a}_d(b+\frac{\sum_{m=1}^{d-1}e_mp_m-p_d}{q})+\sum_{m=1}^{d-1}(\hat{a}_m+e_m\hat{a}_d)(x_m-\frac{p_m}{q})|}{\hat{q}^{-1}H(\hat{P})}$$
$$\leq\frac{d\sigma c}{H_{n+k}}=\rho_0R^{-n-k}.$$

□

4.6. **From Homogeneous to Inhomogeneous.** .

To prove the inhomogeneous case, we first introduce some new notations: $c'=\frac{1}{6}cR^{-2}$ and $H'_n=3c'\rho_0^{-1}R^n$.Let $\theta=(\gamma_1,...,\gamma_d)$

$$\mathcal{V}:=\{P\in\mathbb{Q}^d:B_0\cap\Delta(P)\neq\emptyset,|b+\frac{\sum_{k=1}^{d-1}e_k(p_k+\gamma_k)-p_d-\gamma_d}{q}|<\frac{\sigma c'}{q^{1+l}}\}$$

$$\mathcal{V}_n=\{P\in\mathcal{V}:H'_n\leq q^{1+i}<H'_{n+1}\}$$

and

$$\Delta_\theta(P)=\{x\in\mathbb{R}^{d-1}:|x_m-\frac{p_m+\gamma_m}{q}|<\frac{c'}{q^{1+i}},\forall1\leq m\leq d-1\}$$

Proposition 4.3. *If $diam(B)\leq\rho_0R^{-n+1}$ and $\forall k<n,\forall P\in\mathcal{P}_k,B\cap\Delta(P)=\emptyset$, then there is at most one $v\in\mathcal{V}_n$ such that $\Delta_\theta(v)\cap B\neq\emptyset$. Moreover, the corresponding $\Delta_\theta(v)$ can be contained in a hyperplane neighborhood $G(B)^{\frac{1}{3}\rho_0R^{-n}}$.*

Proof. Given such B, and suppose there are two $v_1, v_2 \in \mathcal{V}_n$ such that $\Delta(v_s) \cap B \neq \emptyset, s = 1, 2$. We let $x_s \in \Delta_\theta(v_s) \cap B$ and WLOG, $q_1 \geq q_2$ Then by assumption we have

$$|q_s x_{m,s} - (p_{m,s} + \gamma_m)| \leq \frac{c'}{q_s^i}, s = 1, 2, m = 1, 2, ..., d-1,$$

$$|bq_s + \sum_{m=1}^{d-1} e_m(p_{m,s} + \gamma_m) - p_{d,s} - \gamma_d| \leq \frac{\sigma c'}{q_s^l}.$$

Thus,

$$\begin{aligned} |(q_1 - q_2)x_{m,1} - (p_{m,1} - p_{m,2})| &\leq q_2|x_{m,1} - x_{m,2}| + |q_1 x_{m,1} - p_{m,1} - \gamma_m| \\ &\quad + |q_2 x_{m,2} - p_{m,2} - \gamma_m| \\ &\leq q_2 \rho_0 R^{-n+1} + \frac{2c'}{q^i}. \end{aligned} \tag{4.5}$$

And

$$|b(q_1 - q_2) + \sum_{m=1}^{d-1} e_m(p_{m,1} - p_{m,2}) - (p_{d,1} - p_{d,2})| \leq \frac{2\sigma c'}{q_2^l} \tag{4.6}$$

First, we show that $q_1 = q_2$. Suppose not then

$$\begin{aligned} |x_{m,1} - \frac{p_{m,1} - p_{m,2}}{q_1 - q_2}| &\leq \frac{1}{q_1 - q_2}(q_2 \rho_0 R^{-n+1} + \frac{2c'}{q_2^i}) \\ &\leq \frac{1}{(q_1 - q_2)^{1+i}}(q_1^i q_2 \rho_0 R^{-n+1} + 2c'R) \\ &\leq \frac{1}{(q_1 - q_2)^{1+i}}(\frac{c}{2} + \frac{1}{3}cR^{-1}) \\ &\leq \frac{c}{(q_1 - q_2)^{1+i}}. \end{aligned} \tag{4.7}$$

And similarly,

$$\begin{aligned} |b + \frac{\sum_{m=1}^{d-1} e_m(p_{m,1} - p_{m,2}) - (p_{d,1} - p_{d,2})}{q_1 - q_2}| &\leq \frac{1}{q_1 - q_2}(\frac{2\sigma c'}{q_2^l}) \\ &\leq \frac{\sigma c}{3R(q_1 - q_2)^{1+l}} \leq \frac{\sigma c}{(q_1 - q_2)^{1+l}}. \end{aligned} \tag{4.8}$$

Thus, let $P_0 = (\frac{p_{i,1} - p_{i,2}}{q_1 - q_2})_{i=1,\ldots,d-1}$, then $x_1 \in \Delta(P_0)$. Moreover, notice that $P_0 \in \mathcal{P}$ by (4.8) we can denote n_0 such that $P_0 \in \mathcal{P}_{n_0}$ and by assumption $n_0 \geq n$, otherwise $\Delta(P_0) \cap B \neq \emptyset$ violates the assumption stated in the theorem. Now

$$(q_1 - q_2)^{1+i} \geq \frac{H_{n_0}}{\sigma} \geq \frac{H_n}{\sigma} = dc\rho_0^{-1} R^n.$$

However, it contradicts the fact that

$$(q_1 - q_2)^{1+i} \leq q_1^{1+i} \leq H'_{n+1} = \frac{1}{2} c \rho_0^{-1} R^{n-1}.$$

Therefore, $q_1 = q_2$. Moreover, by (4.5) we have that

$$|p_{i,1} - p_{i,2}| \le q_2\rho_0 R^{-n+1} + \frac{2c'}{q^{r_i}} \le c + 2c' < 3c < 1.$$

Thus, $p_{i,1} = p_{i,2}, i = 1, ..., d-1$ and together with (4.6) we get $p_{d,1} = p_{d,2}$,so $v_1 = v_2$.

The second part of the theorem is true since $\Delta_\theta(v)$ is a box and contained in a hyperplane neighborhood $G(B)^{\frac{1}{3}\rho_0 R^{-n}}$. □

4.7. **Proof of Theorem 2.2.** .

Let $i(n)$ be the smallest number of round such that $\beta R^{-n}\rho_0 < |B_{i(n)}| \le R^{-n}\rho_0$. Then player A includes $\{E_k(B_{i(n)})^{\rho_0 R^{-n-k}}\}_{k\in\mathbb{N}}$ into his $i(n)$-th hyperplane neighborhood collection, where $E_k(B_{i(n)})$ is as described in Corollary 4.2.

Let $B'_{i(n)} = B_{i(n)} - \cup_{k=1}^{n} \cup_{P\in\mathcal{P}_k} \Delta(P)$, then By Proposition 4.3 $\forall n, \exists G_n, s.t. \cup_{v\in\mathcal{V}_{n+1}} \Delta_\theta(v) \cap B'_{i(n)} \in G_n^{\frac{1}{3}\rho_0 R^{-n-1}}$. Thus, we add this hyperplane neighborhood to the neighborhoods Alice chosen in the $i(n)$-th round. It is a legal move because

$$(\frac{1}{3}\rho_0 R^{-n-1})^\gamma + \sum_{k=1}^{\infty}(\rho_0 R^{-n-k})^\gamma < 2\sum_{k=1}^{\infty}(\rho_0 R^{-n-k})^\gamma < (\beta\rho_i)^\gamma.$$

Now

$$\begin{aligned}\cap_{i\in\mathbb{N}} B_i \subset\ & \mathbf{Bad}^L(i,...,i,l) \cap \mathbf{Bad}^L_\Theta(i,...,i,l) \cup (\cup_{n=1}^{\infty}(\cup_{v\in\mathcal{V}_{n+1}}\Delta_\Theta(v) \cap B'_{i(n)}) \\ & \cup (\cup_{k=1}^{\infty} \cup_{P\in\mathcal{C}_{n+k,k}} \Delta(P) \cap B_{i(n)})) \\ & \subset \mathbf{Bad}^L(i,...,i,l) \cap \mathbf{Bad}^L_\Theta(i,...,i,l) \cup (\cup_{n=1}^{\infty} G_n^{\frac{1}{3}\rho_0 R^{-n}} \cup \cup_{k=1}^{\infty} E_k(B_{i(n)})^{\rho_0 R^{-n-k}}).\end{aligned} \tag{4.9}$$

Thus, $\mathbf{Bad}^L(i,...,i,l) \cap \mathbf{Bad}^L_\Theta(i,...,i,l)$ is HPW and so $\mathbf{Bad}^L_\Theta(i,...,i,l)$ containing this set is also HPW and by Proposition 3.1 HAW. □

5. Proof of Theorem 2.3

5.1. **Preliminaries.** .

In view of [6, Lemmata 1.2-1.4], to prove Theorem 2.3 it suffices to show the following theorem:

Theorem 5.1. *Let* $\mathbf{r}$ *be* n *tuple of weights, and* $\varphi : U \subset \mathbb{R} \to \mathbb{R}^n$*, be a map defined on an open interval. Suppose* μ *be a* (C,α)*-Ahlfors regular measure such that* $U \cap \operatorname{supp}\mu \ne \emptyset$ *and* φ *be an analytic map that is nondegenerate. Then*

$$\varphi^{-1}(\boldsymbol{Bad}_\theta(r) \cap \boldsymbol{Bad}_0(r)) \bigcap \operatorname{supp}\mu \ne \emptyset. \tag{5.1}$$

The goal of this section would be to prove Theorem 5.1. The definition of Ahlfors regular measure can be found in [6, Section 1]. Also, this definition would only appear when stating results from [6].

We then give the definition of generalized Cantor set.

Definition 5.1. *Given $R \in \mathbb{Z}^+$, $I \subset \mathbb{R}$ a closed interval, the R-partition of I is the collection of closed intervals obtained by dividing I to R closed subintervals of the same length $R^{-1}|I|$, denoted as $\boldsymbol{Par}_R(I)$.*

Also, given $\mathcal{J}$ a collection of closed intervals,

$$\boldsymbol{Par}_R(\mathcal{J}) := \cup_{I\in\mathcal{J}} \boldsymbol{Par}_R(I).$$

To define a generalized Cantor set, we first let $\mathcal{J} = \{I_0\}$, where I_0 is a closed interval. Then inductively $\mathcal{I}_{q+1} = \mathbf{Par}_R(\mathcal{J}_q)$. In addition, we remove a subcollection $\hat{\mathcal{J}}_q$ from $\mathcal{I}_{q+1}$. In other words, we let $\mathcal{J}_{q+1} = \mathcal{I}_{q+1} \setminus \hat{\mathcal{J}}_q$.

Definition 5.2.

$$\mathcal{K}_\infty := \bigcap_{q\in\mathbb{N}} \bigcup_{I\in\mathcal{J}_q} I$$

is a generalized Cantor set determined by the process above.

Finally, we include a result in [4] to help us get the non-emptiness of a generalized Cantor set.

Theorem 5.2 ([4])**.** *If $\forall q \in \mathbb{N}$, $\hat{\mathcal{J}}_q$ can be written as $\hat{\mathcal{J}}_q = \cup_{p=0}^{q} \hat{\mathcal{J}}_{p,q}$, and we denote $h_{p,q} := \max_{J\in\mathcal{J}_p} \#\{I \in \hat{\mathcal{J}}_{p,q} : I \subset J\}$, such that inductively*

$$t_q := R - h_{q,q} - \sum_{j=1}^{q} \frac{h_{q-j,q}}{\prod_{i=1}^{j} t_{q-i}} > 0, \tag{5.2}$$

then $\mathcal{K}_\infty \neq \emptyset$.

5.2. **Notations and Results from [6].** .

Without loss of generality, $r_1 \geq r_2 \geq ... \geq r_n > 0$.

Let ρ_0 as defined as [6, (1.4)] and I_0 as in [6] satisfying $3^{n+1} I_0 \subset U$ and $3|I_0| \leq \rho_0$.

$$a(t) := \mathrm{diag}\{e^t, e^{-r_1 t}, ..., e^{-r_n t}\}, b(t) := \mathrm{diag}\{e^{-t/n}, e^t, e^{-t/n} ..., e^{-t/n}\}, u(\mathbf{x}) := \begin{pmatrix} 1 & \mathbf{x} \\ 0 & I_n \end{pmatrix}, \tag{5.3}$$

$$z(x) := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ & 1 & \varphi_2'(x) & \cdots & \varphi_n'(x) \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}. \tag{5.4}$$

Let $\beta, \beta' > 1$ such that

$$e^{(1+r_1)\beta} = R = e^{(1+1/n)\beta'}. \tag{5.5}$$

In [6], a non-empty generalized Cantor set $\mathcal{K}_\infty$ determined by $\hat{\mathcal{J}}_q = \cup_{p=0}^{q} \hat{\mathcal{J}}_{p,q}$ has already been constructed within $\varphi^{-1}(\mathbf{Bad}(\mathbf{r})) \cap \operatorname{supp} \mu$. In particular,

$$\hat{\mathcal{J}}_{q,q} := \{I \in \mathcal{I}_{q+1}, \mu(I) < (3C)^{-1}|I|^\alpha\}. \tag{5.6}$$

Specific definition of other parts in this partition is not involved in the proof of Theorem 5.1. Also, denote $h_{p,q} := \max_{J\in\mathcal{J}_p} \#\{I \in \hat{\mathcal{J}}_{p,q} : I \subset J\}, 0 \le p \le q$, [6] provides upper bounds for $h_{p,q}$.

Proposition 5.1. *Let C, α be as defined in Theorem 5.1 and $R^\alpha \ge 21C^2$*

$$h_{q,q} \le R - (4C)^{-2}R^\alpha. \tag{5.7}$$

Proposition 5.2. *There exist constants $R_0 \ge 1, C_0 > 0$ and $\eta_0 > 0$ such that if $R \ge R_0$*

$$h_{p,q} \le C_0 R^{\alpha(1-\eta_0)(q-p+1)}, 0 \le p < q. \tag{5.8}$$

5.3. **New idea.** In [6], a non-empty generalized Cantor set inside the intersection of the support of the measure and the preimage of a homogeneous bad set was constructed. Now, based on this and Theorem 5.3, to get a non-empty set that is also inside the preimage of inhomogeneous bad set, we would only have to further remove the inhomogeneous "dangerous set" with respect to one integer vector, rather than all of them as done for the homogeneous case in [6]. The good thing about this, is that we don't need the inhomogeneous version of quantitative non-divergence estimates, which for now we don't have in hand.

5.4. **From Homogeneous to Inhomogeneous.** .

It can be checked that

$$\begin{gathered}\mathbf{x} \in \mathbf{Bad}_\theta(\mathbf{r}) \iff \exists k > 0, b > 1, s.t. \forall a_0 \in Z, \mathbf{a} \in Z^d, \\ \textbf{we cannot find } t \in N, s.t. |a_0 + \mathbf{a}\cdot\mathbf{x} + \theta| \le kb^{-t}, |a_i| \le b^{r_i t}, \forall 1 \le i \le d.\end{gathered} \tag{5.9}$$

Let $f_0 := \max_{x\in I_0, i=1,2,\dots,d} |\varphi_i'(x)|$ which makes sense since φ_i' is continuous.

Moreover, we can find $\xi > 0$, s.t $\forall \mathbf{y}$ with $\|\mathbf{y}\| \ge e^{-\epsilon\beta}$, we have $\|u_1(O(1))^{-1}b(-\beta')\cdot \mathbf{y}\| \ge \xi$. In particular, according the result proven in [6, Equation 2.34],

$$\forall q \in \mathbb{N}, \forall x \in J \in \mathcal{J}_q, \forall\ 8 \le q_0 \le q, a(\beta q_0)u(\varphi(x))\mathbb{Z}^{n+1} \in K_\xi. \tag{5.10}$$

.

Let β, β' be as in (5.5). We define the following constants:

$$\begin{gathered}\lambda_1 := \lceil \ln \max\{(\frac{2n(n+1)f_0|I_0|}{\xi})^{1/\beta r_1}, (\frac{n+1}{\xi})^{\frac{1}{\beta r_n}}, e^8\}\rceil, \\ k_1 := \frac{\xi}{n+1}e^{-\beta\lambda_1}, \\ k_2 := 2^{-3-1/r_n}k_1, \\ \lambda_2 := \lceil \ln 2^{1/\beta r_n}\rceil, \\ \Omega := \max(\lceil \frac{16r_1}{\gamma\epsilon}\rceil, \lceil \frac{8}{\lambda_1+\lambda_2}\rceil, 2).\end{gathered}$$

Theorem 5.3. *Given $q \in \mathbb{N}$, and an interval $I \in \mathcal{J}_{q+\lambda_1+\lambda_2}$, there is at most one $a_0 \in \mathbb{Z}, \boldsymbol{a} \in \mathbb{Z}^n$ such that $\exists\ x \in I$ with*

$$|a_0 + \boldsymbol{a}\cdot\varphi(x) + \theta| < k_2 e^{-\beta q}, |a_i| < e^{r_i\beta q}, i > 0. \tag{5.11}$$

Proof. Suppose there are two such integer vectors $a_{i,s}, i = 0, 1, ..., d, s = 1, 2.$, and $x_1, x_2 \in I$ are the corresponding numbers that satisfies Equation (5.11).

Now $\forall i > 0, |a_{i,1} - a_{i,2}| < e^{r_i\beta(q+\lambda_2)}$, then $r_i \geq r_n$ and $|a_{i,1} - a_{i,2}| < 2e^{r_i\beta q} \leq e^{r_i\beta(q+\lambda_2)}$ Moreover, by Mean Value Theorem,

$$|a_{0,1} - a_{0,2} + (\mathbf{a}_1 - \mathbf{a}_2)\varphi(x_1)| \leq |\mathbf{a}_2 \cdot (\varphi(x_1) - \varphi(x_2))| + \sum_{s=1,2} |a_{0,s} + \mathbf{a}_s\varphi(x_s) + \theta|$$
$$\leq e^{r_1\beta q} n f_0 |I_0| e^{-(1+r_1)\beta(q+\lambda_2+\lambda_1)} + \frac{1}{4} k_1 e^{-\beta(q+\lambda_2)} < k_1 e^{-\beta(q+\lambda_2)}. \tag{5.12}$$

Thus, we observe that $\mathbf{w} := a(\beta(q+\lambda_1+\lambda_2))u(\varphi(x_1)) \cdot \begin{pmatrix} a_{0,1} - a_{0,2} \\ a_{1,1} - a_{1,2} \\ \vdots \\ a_{d,1} - a_{d,2} \end{pmatrix}$ has norm less than ξ, which together with the assumption that $\begin{pmatrix} a_{0,1} - a_{0,2} \\ a_{1,1} - a_{1,2} \\ \vdots \\ a_{d,1} - a_{d,2} \end{pmatrix}$ is non-zero violates (5.10) and leads to a contradiction. The reason why the norm is small is that by previous calculation $|w_0| \leq k_1 e^{\beta\lambda_1} \leq \frac{\xi}{n+1}$ and $\forall i > 0, |w_i| < e^{-r_i\lambda_2} \leq \frac{\xi}{n+1}$ and the norm is less than $\frac{\xi}{\sqrt{n+1}} \leq \xi$. □

Lemma 5.1. *$|s_0 + \boldsymbol{s} \cdot \varphi(x) + \theta| < k_2 e^{-\beta q}, |s_i| < e^{\beta r_i q}, 1 \leq i \leq n$ implies $\|b(\beta')a(\beta(q+\lambda_1))z(x)u(\varphi^{\theta,s}(x)) \cdot \begin{pmatrix} s_0 \\ s \end{pmatrix}\| < e^{-\epsilon\beta}$.*

Proof. By definition and the last paragraph in the proof of Theorem 5.3, we can see that $|s_0 + \mathbf{s} \cdot \varphi(x) + \theta| < k_2 e^{-\beta q}, |s_i| < e^{\beta r_i q}, 1 \leq i \leq n$ implies $\|a(\beta(q+\lambda_1))u(\varphi^{\theta,s}(x)) \begin{pmatrix} s_0 \\ s \end{pmatrix}\| < \xi$. Now ,$\|b(\beta')a(\beta(q+\lambda_1))u(\varphi^{\theta,s}(x)) \begin{pmatrix} s_0 \\ s \end{pmatrix}\| < e^{-\epsilon\beta}$, otherwise by [6, Equation (2.33)]it violates the definition of ξ. □

First, we introduce some new inhomogeneous notations .

Definition 5.3. *Given the function $\varphi : U \to \mathbb{R}^n, 0 \neq s \in \mathbb{Z}^n, \theta \in \mathbb{R}$ with $j := \min\{i, s_i \neq 0\}$ we define a new function $\varphi^{\theta,s} : U \to R^n$ as follows*

$$\varphi_i^{\theta,s} := \begin{cases} \varphi_j + \dfrac{\theta}{s_j}, i = j, \\ \varphi_i, i \neq j. \end{cases} \tag{5.13}$$

5.5. **The new cantor set $\mathcal{K}'_\infty$.** Now we define a new generalized cantor set in the following way. If $Q \in \mathbb{N}$ and there does not exist $q \in \mathbb{N}$ such that $\Omega(q + \lambda_1 + \lambda_2)$, we define $\mathcal{M}_{p,Q} := \emptyset, \forall 0 \leq p \leq Q$.

For simplicity we denote

$$H^{\theta,s}(x) = H_l^{\theta,s}(x) := b(\beta' l)a(\beta(q+\lambda_1))z(x)u(\varphi^{\theta,s}(x)). \tag{5.14}$$

If not, i.e. $Q = \Omega(q+\lambda_1+\lambda_2)$ for some $q \in \mathbb{N}$, $\mathcal{M}_{Q,Q} := \emptyset$. Meanwhile, define $\mathcal{M}_{0,Q}$ to be the collection of $I \in \mathcal{I}_{Q+1} - \hat{\mathcal{J}}_{Q,Q}$ such that there exists $l \in \mathbb{Z}$ with $\max(1, Q/8) \le l \le Q/4$ satisfying

$$\begin{aligned} &\exists s_0 \in \mathbb{Z}, 0 \neq s \in \mathbb{Z}^n, s.t. |s_0 + \mathbf{s}\cdot\varphi(x) + \theta| < k_2 e^{-\beta q}, |s_i| < e^{\beta r_i q}, 1 \le i \le n \\ &\text{and } \|H_l^{\theta,s}(x)\cdot \begin{pmatrix} s_0 \\ s \end{pmatrix}\| < e^{-\epsilon\beta l}, \text{ for some } x \in I. \end{aligned} \tag{5.15}$$

If $0 < p \le Q/2$ or $0 < p < Q$ with $p \neq Q \mod 4$, as usual we let $\mathcal{M}_{p,Q} := \emptyset$. Finally if $Q/2 < p < Q$ and $p = Q - 4l$ for some $l \in N$, we define

$$\mathcal{M}_{p,Q} := \{I \in \mathcal{I}_{Q+1} - (\hat{\mathcal{J}}_{Q,Q} \cup \cup_{0\le p'<p}\mathcal{M}_{p',Q}) : (5.15) \text{ holds}\}.$$

Remark 5.1. *We use Q that is larger than q, because we want to remove the additional inhomogeneous "dangerous set" a bit later than the original homogeneous "dangerous set", such that the uniqueness as described in Theorem 5.3 is ensured.*

Now, by induction on $q \in \mathbb{N}$ we define $\mathcal{K}'_\infty$ by letting $\hat{\mathcal{J}}'_{p,q} := \mathcal{I}'_{q+1} \cap (\mathcal{M}_{p,q} \cup \hat{\mathcal{J}}_{p,q}), \forall 0 \le p \le q$.

Lemma 5.2.

$$\mathcal{K}'_\infty \subset \mathcal{K}_\infty,$$

where $\mathcal{K}_\infty$ is the cantor set defined in [6, Section 2.5].

Proof. Note that by definition, $\hat{\mathcal{J}}_{p,q} \subset \hat{\mathcal{J}}'_{p,q}, \forall p,q \in \mathbb{N}$, so $\mathcal{J}'_q \subset \mathcal{J}_q, \forall q \in \mathbb{N}$ and $\mathcal{K}'_\infty = \cap_{q\in\mathbb{N}} \cup_{I\in\mathcal{J}'_q} I \subset \cap_{q\in\mathbb{N}} \cup_{I\in\mathcal{J}_q} I = \mathcal{K}_\infty$. ☐

Lemma 5.3. $\mathcal{K}'_\infty \subset \varphi^{-1}(\boldsymbol{Bad}_\theta(r))$.

Proof. Note that $\forall q \ge 1, I \in \mathcal{J}_{\Omega(q+\lambda_1+\lambda_2)}$, since by construction $\Omega(q+\lambda_1+\lambda_2) > 8$, for $l = 1$, (5.15) does not hold. Moreover, by Lemma 5.1, $|s_0 + \mathbf{s}\cdot\varphi(x) + \theta| < k_2 e^{-\beta q}, |s_i| < e^{\beta r_i q}, 1 \le i \le n$ implies $\|b(\beta')a(\beta(q+\lambda_1))z(x)u(\varphi^{\theta,s}(x)) \cdot \begin{pmatrix} s_0 \\ s \end{pmatrix}\| < e^{-\epsilon\beta}$, so when $l = 1$, (5.15) does not hold is equivalent to there does not exist $s_0 \in \mathbb{Z}, 0 \neq s \in \mathbb{Z}^n$ such that $|s_0 + \mathbf{s}\cdot\varphi(x) + \theta| < k_2 e^{-\beta q}, |s_i| < e^{\beta r_i q}, 1 \le i \le n$, shich is exactly the definition of $\mathbf{Bad}_\theta(r)$.

☐

Now by Lemma 5.3 and together with the fact proven in [6, Proposition 2.6] that

$$\mathcal{K}'_\infty \subset \mathcal{K}_\infty \subset \varphi^{-1}(\mathbf{Bad}(r)) \cap \operatorname{supp}\mu,$$

where μ is as in Theorem 5.1. Thus we have

$$\mathcal{K}'_\infty \subset \varphi^{-1}(\mathbf{Bad}_\theta(r)) \cap \varphi^{-1}(\mathbf{Bad}(r)) \cap \operatorname{supp}\mu = \varphi^{-1}(\mathbf{Bad}_\theta(r) \cap \mathbf{Bad}(r)) \cap \operatorname{supp}\mu.$$

Now to prove Theorem 5.1 it is left to show that $\mathcal{K}'_\infty$ is non-empty, which will be the content of next few sections.

5.6. **Upper Bounds for $h'_{p,q}$.** .

Similar to [6], we give an upper bound for $h'_{p,q} := \max_{J\in\mathcal{J}'_p} \#\{I \in \hat{\mathcal{J}}'_{p,q} : I \subset J\}$. Note that $\mathcal{J}'_p \subset \mathcal{J}_p$, while $\#\{I \in \hat{\mathcal{J}}'_{p,q} : I \subset J\} \leq \#\{I \in \hat{\mathcal{J}}_{p,q} : I \subset J\} + \#\{I \in \mathcal{M}_{p,q} : I \subset J\}$, we only have to find an upper bound for $f_{p,q} := \max_{J\in\mathcal{J}_p} \#\{I \in \mathcal{K}_{p,q} : I \subset J\}$.

Proposition 5.3. *Suppose $R^\alpha \geq 21C^2$, then for every $q \geq 0$ we have*

$$h'_{q,q} \leq R - (4C)^{-2}R^\alpha. \tag{5.16}$$

Proof. By definition $f_{q,q} = 0, \forall q \in \mathbb{N}$. Thus, $h'_{q,q} \leq h_{q,q} + f_{q,q}$ still satisfies the upper bound in Equation (5.7). □

The following is analogous to Equation (5.8).

Proposition 5.4. *For $0 < p < q$, there exist constants $R'_2 \geq 1, C'_2 > 0$ and $\eta'_2 > 0$ such that if $R \geq R'_2$,*

$$h'_{p,q} \leq C'_2 R^{\alpha(1-\eta'_2)(q-p+1)}. \tag{5.17}$$

Proof. If $Q \neq \Omega(q + \lambda_1 + \lambda_2), \forall q \in \mathbb{N}$, $f_{p,Q} = 0, \forall p \leq Q$. These are the trivial upper bounds.

If $Q = \Omega(q+\lambda_1+\lambda_2)$ for some $q \in \mathbb{N}$, as defined, $f_{p,Q} = 0$ if $0 < p \leq Q/2$ or $0 < p < Q$ with $p \neq Q \mod 4$. If otherwise, $Q/2 < p < Q$ and $Q = p + 4l$ for some $l \in \mathbb{N}$, and $J \in \mathcal{J}_p$ is a fixed interval, then $p > q+\lambda_1+\lambda_2$ and by Theorem 2.5, s_0, s that make (5.15) hold are unique within J, so it suffices to check this unique integer vector. Now, since l is fixed,Note that the function $\phi(x) = \|H^{\theta,s}(x)\cdot\begin{pmatrix} s_0 \\ s \end{pmatrix}\|$ is (C,γ)-good for some universal positive constants by [6, Corollary 5.16] , and we have If $\sup_{x\in J}\|H^{\theta,s}(x) \cdot \begin{pmatrix} s_0 \\ s \end{pmatrix}\| \geq 1$. Hence , we have that $\mu(\{x \in J, \|H^{\theta,s}(x)\cdot\begin{pmatrix} s_0 \\ s \end{pmatrix}\| < e^{-\epsilon\beta l}\}) \leq C3^{\alpha+\gamma}e^{-\gamma\epsilon\beta l}|I_0|^\alpha R^{-p\alpha}$ and since $I \notin \hat{\mathcal{J}}_{Q,Q}$, we have $\#\{I \in \mathcal{M}_{p,q} : I \subset J\} \leq C3^{\alpha+\gamma}e^{-\gamma\epsilon\beta l}|I_0|^\alpha R^{-p\alpha}C|I_0|^{-\alpha}R^{\alpha(Q+1)}$, thus $h'_{p,Q} \leq h_{p,Q} + f_{p,Q}$ still satisfies the upper bound described in Equation (5.8).

Here we give reasons why the other case is impossible. First, we provide a simplified version of [6, Lemma 3.1].

Lemma 5.4. *There exists R_0 such that $\forall R \geq R_0, 0 \leq l, l' \leq Q/2, x, x_0$ such that $x = x_0 + \theta R^{-Q-1+l'}$ for some $|\theta| \leq |I_0|$ and any $v \in \mathbb{Z}^{n+1}$, we have*

$$\frac{1}{2} \leq \frac{\|u(\theta R^{l'-l})H^{\theta,s}(x_0)v\|}{\|H^{\theta,s}(x)v\|} \leq 2. \tag{5.18}$$

We can borrow this lemma because $z(x)$, i.e. the derivatives of $\varphi(x)$ and $\varphi^{\theta,s}(x)$ are the same, so the calculations can follow the ones in [6] and this lemma holds with respect to $\varphi^{\theta,s}$.

Lemma 5.5. $\sup_{x\in J}\left\|H^{\theta,s}(x)\cdot\begin{pmatrix} s_0 \\ s \end{pmatrix}\right\| < 1$ *is impossible.*

Proof. Suppose not, fix an $x_0 \in J$, and let $\mathbf{b} = H^{\theta,s}(x_0)\begin{pmatrix} s_0 \\ s \end{pmatrix}$. WLOG, we have $x = x_0 + \frac{1}{2}|I_0|R^{-Q-1+l'} \in J$, where $l' = 4l+1$. Then by assumption and Lemma 5.4,

$$\|\mathbf{b} + \frac{1}{2}|I_0|R^{l'-l}\mathbf{e}_1\| \leq \|u(\frac{1}{2}|I_0|R^{l'-l}\mathbf{e}_1)\mathbf{b}\| \leq 2\|H^{\theta,s}(x)\begin{pmatrix} s_0 \\ s \end{pmatrix}\| < 2$$

In particular, $\frac{1}{2}|I_0|R^{l'-l}|b_2| < 2 + |b_1| < 4$, so $|b_2| < \frac{8}{I_0}R^{l-l'} < R^{-3l}$. Then,

$$\|H^{\beta,s}_{2l}(x_0)\begin{pmatrix} s_0 \\ s \end{pmatrix}\| = \|b(\beta' l)\mathbf{b}\| \leq e^{-\beta' l/n}\|\mathbf{b}\| + e^{\beta' l}|b_2| \leq 3e^{-3\epsilon\beta l} < e^{-2\epsilon\beta l}. \tag{5.19}$$

This implies that (5.15) holds for $2l$, but is a contradiction to $I \notin \mathcal{M}_{p',Q}$, where $p' = Q - 8l < p$ if $l < Q/16$ and $p' = 0 < p$ if $l \geq Q/16$. ☐

Thus, by Lemma 5.4, the proof of upper bounds for $h'_{p,q}, 0 < p < q$ is done. ☐

Proposition 5.5. *For $p = 0$, there exist constants $R'_1 \geq 1, C'_1 > 0$ and $\eta'_1 > 0$ such that if $R \geq R'_1$,*

$$h'_{0,q} \leq C'_1 R^{\alpha(1-\eta'_1)(q+1)}. \tag{5.20}$$

Proof. When $p = 0$, for some $l \in N$

$$\{x \in I, (5.15) \text{ holds}\} \subset \cup_{|s_i| < e^{\beta r_i q}}\{x \in I_0, \|H^{\theta,s}(x) \cdot \begin{pmatrix} s_0 \\ s \end{pmatrix}\| < 3e^{-\epsilon\beta l}\}. \tag{5.21}$$

Notice that there are at most $6M_3 n e^{r_1\beta q}$ distinct $\begin{pmatrix} s_0 \\ s \end{pmatrix}$ satisfying (5.15) for fixed θ, where $M_3 := 1 + \sup_{x\in I_0}\|\varphi(x)\|$. Thus, if we let N_l to be the number of intervals in $\mathcal{M}_{0,Q}$ such that (5.15) holds for l. Moreover, by [3, Corollary 4],

$$\rho := \inf_{a_0\in\mathbb{Z}, \mathbf{a}\in\mathbb{Z}^n, \|a\|\geq H_0} \sup_{x\in I_0} |a_0 + \mathbf{a}\cdot\varphi(x) + \theta| > 0.$$

Then again by [6, Corollary 5.16], $N_l \leq 6\rho^{-1}nCM_3 3^\gamma |I_0|^{-\alpha} R^{\frac{r_1}{1+r_1}q + \alpha(Q+1) - \eta'_1 l}$. Thus,

$$f_{0,Q} \leq 36\rho^{-1}nCM_3 3^\gamma |I_0|^{-\alpha} R^{\frac{r_1}{1+r_1}q + \alpha(Q+1) - \eta'_1 l_{min}} \leq 36\rho^{-1}nCM_3 3^\gamma |I_0|^{-\alpha} R^{\alpha(1-\frac{\eta_1}{2})(Q+1)},$$

and conditions in (5.8) still holds for $h'_{0,Q} \leq h_{0,Q} + f_{0,Q}$. ☐

5.7. **Proof of Theorem 5.1.** .

It is time to prove the non-emptiness of $\mathcal{K}'_\infty$. By Theorem 5.2, it suffices to prove inductively that

$$t'_q := R - h'_{q,q} - \sum_{j=1}^{q} \frac{h'_{q-j,q}}{\prod_{i=1}^{j} t'_{q-i}} \geq (6C)^{-2}R^\alpha. \tag{5.22}$$

Now by Equation (5.17) and Equation (5.20), we have that

$$h'_{p,q} \leq C'_0 R^{\alpha(1-\eta'_0)(q-p+1)}, 0 \leq p < q$$

for some positive constants C_0', η_0'. Then together with Equation (5.16) and suppose (5.22) holds for $q < q_1$, we have

$$t'_{q_1} \geq (4C)^{-2} - R^{\alpha}(C_0' R^{-\eta_0'\alpha} \sum_{j=1}^{\infty} (\frac{36C^2}{R^{\eta_0'\alpha}})^j).$$

Thus, we only have to make sure that R is large enough such that $C_0' R^{-\eta_0'\alpha} < \frac{1}{32C^2}$ and $\sum_{j=1}^{\infty} (\frac{36C^2}{R^{\eta_0'\alpha}})^j \leq 1$ to make the induction work. Thus the proof is done.

6. Final Comments and Further Work

These kinds of extensions from homogeneous to inhomogeneous largely depend on the techniques used in the proof of homogeneous results. In other words, the methods could for now only extend the homogeneous results to simultaneous inhomogeneous if they are proved via the simultaneous homogeneous bad, and vice versa for dual bad. The reasons behind it could be that we are in the lack of inhomogeneous versions of corresponding techniques such as simplex lemma or quantitative non-divergence estimates.

There are many interesting questions in both dual inhomogeneous bad and simultaneous inhomogeneous bad, and their intersections. For instance, extending the winning property of weighted bad in $\mathbb{R}^n$ to inhomogeneous, which in this REU project we didn't have time to figure out.

Dream theorem 6.1. *$\forall \theta \in \mathbb{R}, r \in \mathbb{R}^n$ with $r_i \geq 0, \sum_i r_i = 1$, $\boldsymbol{Bad}_{\theta}(r)$ is HAW.*

Dream theorem 6.2. *$\forall \Theta \in \mathbb{R}^n, r \in \mathbb{R}^n$ with $r_i \geq 0, \sum_i r_i = 1$, $\boldsymbol{Bad}_{\Theta}(r)$ is HAW.*

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