EKEDAHL-OORT TYPES OF ARTIN-SCHREIER CURVES

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Abstract. Let k be an algebraically closed field of characteristic $p > 0$. We compute three different representations of the Ekedahl–Oort type (i.e. the isomorphism type of the p-kernel group scheme $J[p]$) for Artin Schreier curves (i.e. k-curves defined by $y^p - y = f(x)$ for $f(x) \in k(x)$) when $f(x) \in k[x]$. We start by computing the Hasse-Witt triple (Q, Φ, Ψ); we proceed to find the corresponding polarized Dieudonne Module (M, F, V, b); and we conclude by providing an algorithm for computing the final type when $f(x) = x^m$ for nonnegative integer m, which is canonical in the sense that it doesn't depend on a chosen basis. An implementation of the algorithms is available in Python.

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Date: July 12, 2023.

1. Introduction

Throughout the project, we will assume that the base field $k = \overline{k}$, $\mathrm{char}(k) = p \neq 0$, and let $\sigma : k \to k, a \mapsto a^p$ be the Frobenius automorphism.

1.1. The Ekedahl–Oort Type.

Let C/k be a smooth projective curve. Associated to C is its Jacobian variety $J(C)$. Since the Jacobian of a smooth curve C/k is a group scheme, it is natural to investigate the following question:

Question 1.1. Which group schemes arise as the p-torsion $(J(C)[p])$ for a curve?

By Dieudonne theory, $J[p]$ is equivalent to a triple (M, F, V) where

(1) M is a finite dimensional k-vector space.
(2) $F : M \to M$ is σ-linear (i.e. $F(ax) = \sigma(a)F(x)$)
(3) $V : M \to M$ is σ^{-1}-linear (i.e. $F(ax) = \sigma^{-1}(a)F(x)$)
(4) $\ker F = \mathrm{Im} V$, $\mathrm{Im} F = \ker V$
(5) b: alternating bilinear form induced by the polarization map $J[p] \to J[p]^\vee$,

The quadruple (M, F, V, b) is called a *polarized Dieudonne module*. The isomorphism class of polarized Dieudonne modules is defined as the *Ekedahl–Oort type* of a curve. In particular, the polarized Dieudonne module can be encoded and recovered with a numerical invariant called the *final type*, whose advantage is that it is canonical in the sense that it doesn't involve a choice for basis, and its construction will be discussed in Section 4. Now question 1.1 becomes:

Question 1.2. Which Ekedahl-Oort types arise from Jacobian of curves?

The Ekedahk-Oort type can be equivalently characterized by the Hasse-Witt triple (Q, Φ, Ψ) [Moo22, Theorem 2.8] where

(1) Q is a finite dimensional k-vector space.
(2) $\Phi : Q \to Q$ is σ-linear
(3) $\Psi : \ker \Phi \to \mathrm{Im}\Phi^\perp = \{\lambda \in Q^\vee | \lambda(q) = 0, q \in \mathrm{Im}(\Phi)\}$ is σ-linear bijection.

When C/k is a smooth proper curve, $Q = H^1(C, \mathcal{O}_C)$ and Φ is the frobenius on $H^1(C, \mathcal{O}_C)$ [Oda69, Section 5].

There is currently no known algorithm to compute the Ekedahl-Oort type for general curves. Known cases include: Complete intersection curve with $p \nmid \deg C$ ([Moo22]), Hyperelliptic curves in odd characteristic ([DH17]), and Cyclic covers of P^1 whose Galois group has order prime to p ([LMS23]). In this project, we will compute the Ekedahl-Oort type of a class of Artin-Schreier curves, which are defined in Definition 1.1.

Definition 1.1. (Artin-Schreier curves) An Artin-Schreier curve C/k is defined equivalently as

(1) A $\mathbb{Z}/p\mathbb{Z}$ Galois cover of $\mathbb{P}^1$;
(2) The normalization of $\mathbb{P}^1$ inside an extension of function fields $k(x) \hookrightarrow K$, which is Galois with $Gal(K/k(x)) = \mathbb{Z}/p\mathbb{Z}$;
(3) A smooth projective curve whose function field is of the form $K \cong k(x)[y]/(y^p - y - f(x)$ for some $f(x) \in k(x)$.

We will focus on the case when $f(x) \in k[x]$ in this project.

1.2. **Key Results.** Our main goal is to compute all three representations of Ekedahl-Oort type of Artin Schreier curves. In Section 2, we compute the Hasse-Witt Triple for general $f(x) \in k[x]$. Here we illustrate the main result of Section 2 in the special case when $f(x) = x^m$, where m is a nonnegative integer. The general case for $f(x) \in k[x]$ is more notationally involved and omitted here.

Theorem. (Hasse-Witt Triple) Let C/k is an Artin-Schreier curve defined by $y^p - y = f(x)$, and $f(x) = x^m$ for nonnegative integer m. Define

$$Q = k\langle x^i y^j \mid 0 \leq j \leq p-1, -\frac{jm}{p} < i < 0\rangle;$$

Let $x^i y^j$ and $x^{i'} y^{j'}$ be valid elements in the basis of Q given above. Define Φ on a basis and extend linearly: Let the coefficient of $x^{i'} y^{j'}$ in $\Phi(x^i y^j)$ be

$$\begin{cases} \binom{j}{j'} & \text{if } mj - mj' + ip = i' \\ 0 & \text{otherwise;} \end{cases}$$

Let $x^i y^j$ and $y^r x^b \mathrm{d}x$ be valid elements in this basis of Q and $Q^\vee = H^0(C, \Omega^1_C)$ (with a basis given in 2.3) respectively. Then $\ker \Phi$ is only generated by elements given in the basis of Q above. Define Ψ on a basis and extend linearly: If $x^i y^j \in \ker \Phi$, the coefficient of $y^r x^b \mathrm{d}x$ in $\Psi(x^i y^j)$ is

$$-m\binom{j}{r+1}(r+1)\delta_1 + \binom{j}{r}(mj - mr + ip)\delta_2,$$

where

$$\delta_1 := \begin{cases} 1 & \text{if } mj - m(r+1) + ip \geq 0 \text{ and } mj - m(r+1) + ip + m - 1 = b \\ 0 & \text{otherwise} \end{cases}$$

$$\delta_2 := \begin{cases} 1 & \text{if } mj - mr + ip - 1 = b \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then (Q, Φ, Ψ) is a Hasse-Witt Triple associated with the curve C.

In Section 3, we compute the polarized Dieudonne module (M, F, V, b), where M follows immediately from our results in Section 2, b is computed for general $f \in k[x]$, and F, V are computed only for $f \in x^m$. The main results are omitted here for the sake of brevity. Finally, Section 4 provides an algorithm for computing the final type in the special case when $f(x) = x^m$. The Appendix gives a description for the main functions in the python implementation of our results in the special case when $f(x) = x^m$.

1.3. **Notation.** C/k is an Artin-Schreier curve defined by $y^p - y = f(x)$, and $f \in k[x]$ with $m := \deg f$. By [Far09, Prop 2.1.1], we can assume that f is monic with $p \nmid m$. Write

$$f(x) = x^m + a_{m-1}x^{m-1} + \dots + a_0.$$

2. Hasse-Witt Triples

By [Moo22], the Hasse-Witt triple associated to C is (Q, Φ, Ψ), where $Q = H^1(C, O_C)$ is the first cohomology of C, Φ is the Frobenius endomorphism on $H^1(C, O_C)$, and Ψ is a map $\Psi : \ker(\Phi) \to \mathrm{Im}(\Phi)^\perp$, which we will describe in detail in 2.3.1.

2.1. **Hasse-Witt Triple:** Q**.** We begin with a technical lemma which will be useful for computing $Q \cong H^1(C, O_C)$:

Lemma 2.1. *Given $f(x) \in k[x]$ with $m = \deg f$, set $K \coloneqq k(x)[y]/(y^p - y - f(x))$.*

(1) The integral closure of $\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1 \backslash \infty) = k[x]$ in K is $R_1 \coloneqq k[x, y]/(y^p - y - f(x))$.
(2) The integral closure of $\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1 \backslash 0) = k[\frac{1}{x}]$ in K is $R_2 \coloneqq k\langle x^i y^j \mid -ip - jm \geq 0\rangle$.
(3) The integral closure of $\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1 \backslash \{0, \infty\}) = k[x, \frac{1}{x}]$ in K is $R_3 \coloneqq k[x, \frac{1}{x}, y]/(y^p - y - f(x))$.

Proof. (1) We first claim that R_1 is integrally closed. Indeed, the Jacobian matrix of $\mathrm{Spec} k[x, y]/(y^p - y - f(x))$ is

$$Jac = \begin{pmatrix} -f'(x) & -1 \end{pmatrix},$$

which has corank 1, and this is equal to the pure dimension of $\mathrm{Spec} R_1 = \mathrm{Spec} k[x, y]/(y^p - y - f(x))$ at any closed point. By the Jacobian criterion, $\mathrm{Spec} R_1$ is smooth, hence R_1 is normal. Since there are no intermediate fields between $k(x)$ and K, the fraction field of $k[x, y]/(y^p - y - f(x))$ is K, and therefore it is integrally closed in K.
We then show it is the integral closure of $k[x]$ in K. If t is in the integral closure of $k[x]$ in K, then t is integral over $k[x]$, so it is integral over R_1 and therefore it must be in R_1. On the other hand, R is contained in the integral closure of $k[x]$ in K, since x and y are both integral over $k[x]$, and they generate R as a k-algebra.

(2) The integral closure of $R \coloneqq k[\frac{1}{x}]$ in K is equal to

$$\cap_{R \subseteq O \subseteq K} O = \cap_{\text{places of } K} O = \cap_{\text{closed points } q \text{ of } C} O_q,$$

where O is the set of all valuation rings in K that contain R, and $O_q = \{g \in K \mid v_q(g) \geq 0\}$. Therefore, it suffices to find the set of elements $g \in K$ that have $v_q(g) \geq 0$ at all closed points $q \in C$.
Suppose the cover $\pi : C \to \mathbb{P}^1$ sends $\tilde{q}$ to q. By the valuation formula,

$$v_{\tilde{q}}(g) = e(\tilde{q} \mid q) v_q(g),$$

where $e(\tilde{q}, q)$ is the ramification index.

- If $q = \tilde{\infty}$, π is totally ramified at $\tilde{q}$; by [Sti09, Prop.3.7.8], $e(\tilde{q} \mid q) = p$. Since $v_\infty(x) = -1$, $v_{\tilde{\infty}}(x) = -p$. Also, $v_{\tilde{\infty}}(f(x)) = v_{\tilde{\infty}}(y^p - y)$, which gives $v_{\tilde{\infty}}(y) = -m$ (where $m = \deg f$).
- If $q \neq \tilde{\infty}$ is a closed point of C, π is unramified at $\tilde{q}$. Again by [Sti09, Prop.3.7.8], thus $e(\tilde{q} \mid q) = 1$ and $v_{\tilde{q}}(g) = v_q(g)$. But then $v_q(x) \geq 0$ and $v_q(y) \geq 0$, so $v_{\tilde{q}}(x^i y^j) = v_q(x^i y^j) \geq 0$ all the time.

Therefore,

$$\cap_{\text{closed points } q \text{ of } C} O_q = O_{\tilde{\infty}} = k\langle x^i y^j \mid -ip - jm \geq 0\rangle$$

(3) We first claim that $R_3 = k[x, \frac{1}{x}, y]/(y^p - y - f(x))$ is integrally closed: Observe that $R_3 = k[x, \frac{1}{x}, y]/(y^p - y - f(x)) \cong k[x, y, z]/(y^p - y - f(x), xz - 1)$. Therefore the Jacobian of $\mathrm{Spec} R_3$ is given by

$$Jac = \begin{pmatrix} -f'(x) & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

which has corank 2, equivalent to the pure dimension of $\mathrm{Spec} k[x, y, z]/(y^p - y - f(x), xz - 1)$. By an argument similar to (1) (i.e. replacing $k[x]$ with $k[x, \frac{1}{x}]$ everywhere), $k[x, \frac{1}{x}, y]/(y^p - y - f(x))$ is the integral closure of $k[x, \frac{1}{x}]$ in K as well.

□

Now we proceed to find a basis for Q, which is equal to $H^1(C, \mathcal{O}_C)$ as is discussed in 1.1.

Theorem 2.2.

$$H^1(C, \mathcal{O}_C) = k\langle x^i y^j \mid 0 \le j \le p-1, -\frac{jm}{p} < i < 0\rangle.$$

Proof. Let $\pi : C \to \mathbb{P}^1$ be the cover map in Definition 1.1. Consider the natural affine cover of the projective line, $\{\mathbb{P}^1_k \backslash 0, \mathbb{P}^1_k \backslash \infty\}$: We can obtain an affine open cover of C by pulling back via π, i.e. by setting $U_0 \coloneqq \pi^{-1}(\mathbb{P}^1_k \backslash 0)$ and $U_\infty \coloneqq \pi^{-1}(\mathbb{P}^1_k \backslash \infty)$. The Cech complex is given by:

$$0 \to \mathcal{O}_C(U_0) \times \mathcal{O}_C(U_\infty) \xrightarrow{\delta^1} \mathcal{O}_C(U_0 \cap U_\infty) \to 0$$

Since C is the normalization of $\mathbb{P}^1$ in k, $\mathcal{O}_C(\pi^{-1}(U))$ is the integral closure of $\mathcal{O}_{\mathbb{P}^1}(U)$ in $K \coloneqq k(x)[y]/(y^p - y - f(x))$. By Lemma 2.1,

$$\begin{aligned}
\mathcal{O}_C(U_0) &= k[x, y]/(y^p - y - f(x)) \\
\mathcal{O}_C(U_\infty) &= k\langle x^i y^j \mid -ip - jm \ge 0\rangle \\
\mathcal{O}_C(U_0 \cap U_\infty) &= k[x, \frac{1}{x}, y]/(y^p - y - f(x)) \\
&= k\langle x^i y^j \mid i \in \mathbb{Z}, 0 \le j \le p-1\rangle
\end{aligned}$$

The only generators that are not in the image of δ^1 are

$$\{x^i y^j \mid 0 \le j \le p-1, -\frac{jm}{p} < i < 0\},$$

which is a basis for $H^1(C, \mathcal{O}_C)$. □

2.2. **Hasse-Witt Triple: Φ.** As the second step to compute the Hasse-Witt Triple, we give the formula for Φ. Recall from 1.1 that Φ is the induced map of the frobenius on $H^1(C, \mathcal{O}_C)$.

Theorem 2.3. *Suppose $x^i y^j$ and $x^{i'} y^{j'}$ are valid basis elements as in Theorem 2.2. For $f(x) \in k[x]$, the coefficient of $x^{i'} y^{j'}$ in $\Phi(x^i y^j)$ is*

$$\begin{cases} \binom{j}{j'} \sum_{t_0, \ldots t_m} \frac{(j-j')!}{t_m! t_{m-1}! \ldots t_0!} a_{m-1}^{t_{m-1}} \ldots a_0^{t_0} & \text{where } (t_0, \ldots, t_m) \text{ goes over all } m+1 \text{ tuples in } \mathbb{Z}_{\ge 0}^{m+1} \text{ that satisfy } (\star) \\ 0 & \text{if no such } (t_0, \ldots, t_m) \text{ satisfying } (\star) \text{ exists} \end{cases}$$

where

$$(\star) : \begin{cases} t_m + \ldots + t_0 = j - j' \\ m t_m + \ldots + t_1 = i' - ip \end{cases}$$

Proof. By the equivalence relation,

$$\begin{aligned}
(x^i y^j)^p &= x^{ip}(y + f(x))^j \\
&= x^{ip} \sum_{n=0}^{j} \binom{j}{n} y^n (f(x))^{j-n}.
\end{aligned}$$

It suffices to find the coefficient of $x^{i'-ip}$ in $\binom{j}{j'}(f(x))^{j-j'}$, which is given by the multinomial coefficient given in the theorem statement. □

Remark 2.4. A simpler formula for Φ in the special case when $f(x) = x^m$ will be given in 2.4.

2.3. **Hasse-Witt Triple: Ψ.** Recall that

$$\Psi : \ker \Phi \to \mathrm{Im}\Phi^{\perp} = \{\lambda \in Q^{\vee} | \lambda(q) = 0, q \in \mathrm{Im}(\Phi)\}.$$

We will begin this section with an explicit description of Ψ with our open cover, and then proceed to its formula.

2.3.1. *A description of* Ψ. We provide the following description of Ψ given in [LMS23, Section 4.0.1]: Let $\alpha \in \ker \Phi \subseteq H^1(C, O_C)$. Since $\alpha^p = 0 \in H^1(C, \mathcal{O}_C)$, there exists some $\alpha_0 \in \mathcal{O}_C(U_0)$ and $\alpha_\infty \in \mathcal{O}_C(U_\infty)$ such that

$$\alpha^p = \alpha_0 \mid_{U_0 \cap U_\infty} - \alpha_\infty \mid_{U_0 \cap U_\infty} .$$

Taking differentials on both sides, $\mathrm{d}\alpha_0 = \mathrm{d}\alpha_\infty$ on $U_0 \cap U_\infty$, so they glue to some $\omega_\alpha \in Q^{\vee} \cong H^0(C, \Omega^1_C)$. We have $\Psi(\alpha) = \omega_\alpha$.

Remark 2.5. This description relies heavily on the fact of having only two opens in the cover.

2.3.2. *Computing* Ψ. Since $\ker \Phi$ is not necessarily generated only by the basis elements of Q we've computed in Theorem 2.2, we need to find a way to work around that. For $v \in \ker \Phi$, write $v = \sum a_{i,j} x^i y^j$. Then

$$\Phi(v) = \sum a_{i,j}^p \Phi(x^i y^j).$$

By 2.3.1, it suffices to find the terms in $\Phi(v)$ with nonnegative powers for x and taking its differential. Since the differential operator is linear, it suffices to compute $\mathrm{d}\Phi(x^i y^j)$ for each $x^i y^j$ and extract the terms that satisfy the condition. We can thus define $\psi : Q \to Q^{\vee}$ such that

$$\Psi(v) = \sum a_{i,j}^p \psi(x^i y^j),$$

In Theorem 2.7, we will characterize ψ on a basis as a linear combination of a basis of $H^0(C, \Omega^1_C)$ given by

$$\{y^r x^b \mathrm{d}x : 0 \leq r \leq p-2, 0 \leq b \leq m-2, rm + bp \leq pm - m - p - 1\}$$

in [Far09, Prop. 2.2.4].

Remark 2.6. ψ and Ψ are distinct maps. In particular, the definition of ψ on this basis of Q doesn't guarantee that Ψ is necessarily defined for any of these basis elements.

Theorem 2.7. *Suppose $x^i y^j$ and $y^r x^b \mathrm{d}x$ are valid basis elements of Q and $H^0(C, \Omega^1_C)$ respectively. For $f(x) \in k[x]$, the coefficient of $y^r x^b \mathrm{d}x$ in $\psi(x^i y^j)$ is*

$$\begin{cases} \binom{j}{r}(j-r) \sum\limits_{0 \leq s \leq m-1} a_{s+1} \sum\limits_{t_0^s, \dots t_m^s} \frac{(j-r-1)!}{t_m^s! t_{m-1}^s! \dots t_0^s!} a_{m-1}^{t_{m-1}^s} \dots a_0^{t_0^s} + \\ \qquad \binom{j}{r} ip \sum\limits_{l_0, \dots l_m} \frac{(j-r)!}{l_m! l_{m-1}! \dots l_0!} a_{m-1}^{l_{m-1}} \dots a_0^{l_0} - (r+1) \sum\limits_{t \geq 0} B_t (b-t+1) a_{b-t+1} \\ \qquad \textit{where } (s, t_0^s, \dots, t_m^s) \textit{ satisfies } (\star),\ (l_0, \dots, l_m) \textit{ satisfies } (\star) \\ 0 \qquad \textit{if no such tuple satisfying } (\star), (\star) \textit{ exists} \end{cases}$$

where

$$B_t = \begin{cases} \binom{j}{r+1} \sum\limits_{t_0, \dots t_m} \frac{(j-r-1)!}{t_m! t_{m-1}! \dots t_0!} a_{m-1}^{t_{m-1}} \dots a_0^{t_0} \textit{ with } (t_0, \dots t_m) \textit{ that satisfy } (\star) \\ 0 \textit{ if } t < 0 \textit{ or } t > mj - mn + ip \end{cases}$$

$$(\star) : \begin{cases} t_m + ... + t_0 = j - r - 1 \\ mt_m + ... + t_1 = t - ip \end{cases}$$

$$(\star) : \begin{cases} \sum_{i=0}^{m} t_i^s = j - r - 1 \\ s + mt_m^s + ... + t_1^s = b - ip \end{cases}$$

$$(\star) : \begin{cases} \sum_{i=0}^{m} l_i = j - r \\ ml_m + ... + l_1 = b - ip + 1 \end{cases}$$

Proof. Recall that

$$\begin{aligned}(x^i y^j)^p &= x^{ip}(y + f(x))^j \\ &= x^{ip} \sum_{n=0}^{j} \binom{j}{n} y^n (f(x))^{j-n}.\end{aligned}$$

If $y^r x^b \mathrm{d}x$ is a valid basis element, the only place it could occur is when $n = r, r+1$. So it suffices to find the coefficient of x^b in

$$\binom{j}{r}(j-r)(f(x))^{j-r-1}f'(x) + \binom{j}{r}(f(x))^{j-r}ipx^{ip-1} - (r+1)\sum_{t\geq 0} B_t f'(x)x^t,$$

where B_t is the coefficient of x^t in $\binom{j}{r+1}(f(x))^{j-r-1}x^{ip}$. This is given by the sum of multinomial coefficients in the theorem statement. □

Remark 2.8. A simpler formula for Ψ in the special case when $f(x) = x^m$ will be given in 2.4.

2.4. **Special Case:** $f(x) = x^m$**.** When the Artin-Schreier curve is defined by a general polynomial $f(x) \in k[x]$, the formulae for Φ and Ψ are rather complicated. We will conclude this section with simplified formulae and their properties for the special case when $f(x) = x^m$ for positive integer m. In particular, we can explicitly find the kernel of Φ, or the domain of Ψ, in this case.

Proposition 2.9 gives a simple description of Φ in the special case.

Proposition 2.9. *Suppose $x^i y^j$ and $x^{i'} y^{j'}$ are valid basis elements as in Theorem 2.2. If $f(x) = x^m$ for nonnegative integer m, the coefficient of $x^{i'} y^{j'}$ in $\Phi(x^i y^j)$ is*

$$\begin{cases} \binom{j}{j'} & \text{if } mj - mj' + ip = i' \\ 0 & \text{otherwise} \end{cases}$$

Proof. Note that

$$(x^i y^j)^p = x^{ip}(y + x^m)^j = \sum_{n=0}^{j} \binom{j}{n} y^n x^{mj-mn+ip},$$

It suffices to find the coefficient of $x^{i'} y^{j'}$. But the j''s power of y can only occur when $n = j'$. Therefore if $mj - mj' + ip = i'$, the coefficient is $\binom{j}{j'}$; it is 0 otherwise. □

Lemma 2.10 and 2.11 will establish two properties of Φ:

Lemma 2.10. *Suppose $x^i y^j$, $x^{i_1} y^{j_1}$, $x^{i_2} y^{j_2}$ are valid basis elements as in Theorem 2.2. If the coefficients for $x^{i_1} y^{j_1}$, $x^{i_2} y^{j_2}$ are both nonzero in $\Phi(x^i y^j)$, then $i_1 = i_2$, $j_1 = j_2$.*

Proof. In this case,

$$mj - mj_1 + ip = i_1$$
$$mj - mj_2 + ip = i_2$$

We have $m(j_1 - j_2) = i_2 - i_1 < \max\left(\frac{mj_2}{p}, \frac{mj_1}{p}\right)$, thus

$$j_1 - j_2 < \max\left(\frac{j_2}{p}, \frac{j_1}{p}\right) \leq \frac{p-1}{p} \leq 1,$$

so $j_1 = j_2$, $i_1 = i_2$. □

Lemma 2.11. *Suppose $x^i y^j$, $x^{i'} y^{j'}$, $x^{i_1} y^{j_1}$ are valid basis elements as in Theorem 2.2. If the coefficient for $x^{i_1} y^{j_1}$ is nonzero in both $\Phi(x^i y^j)$ and $\Phi(x^{i'} y^{j'})$, then $i = i'$, $j = j'$.*

Proof. We have

$$mj_2 - mj' + i_2 p = i'$$
$$mj_1 - mj' + i_1 p = i'$$

So we have $m(j_2 - j_1) = (i_1 - i_2)p$. As $p \nmid m$, $p \mid j_2 - j_1 \leq p - 1$, which means $j_1 = j_2$, $i_1 = i_2$. □

In particular, we can deduce an explicit description of $\ker \Phi$ from Proposition 2.9 and Lemma 2.11.

Corollary 2.12. *When $f(x) = x^m$ for positive integer m,*

$$\ker \Phi = \langle x^i y^j | mj - mj' + ip \geq 0 \ \textit{or} \ \leq -\frac{j'm}{p} \ \textit{for all}\ 0 \leq j' \leq j \rangle$$

Corollary 2.12 allows us to describe Ψ on the basis of Q given in Theorem 2.2. In this case, ψ as defined in 2.3 agrees with Ψ on the basis given in Corollary 2.12. We conclude this subsection with the formula for Ψ in the special case:

Proposition 2.13. *Suppose $x^i y^j$ and $y^r x^b \mathrm{d}x$ are valid basis elements of Q and $H^0(C, \Omega^1_C)$ respectively, and $x^i y^j \in \ker \Phi$ as in Corollary 2.12. If $f(x) = x^m$ for nonnegative integer m, the coefficient of $y^r x^b \mathrm{d}x$ in $\Psi(x^i y^j)$ is*

$$-m\binom{j}{r+1}(r+1)\delta_1 + \binom{j}{r}(mj - mr + ip)\delta_2,$$

where

$$\delta_1 := \begin{cases} 1 & \text{if } mj - m(r+1) + ip \geq 0 \text{ and } mj - m(r+1) + ip + m - 1 = b \\ 0 & \text{otherwise} \end{cases}$$

$$\delta_2 := \begin{cases} 1 & \text{if } mj - mr + ip - 1 = b \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By 2.3.1, it suffices to find the terms in $(x^i y^j)^p$ with nonnegative powers for x and find the coefficient of $y^r x^b \mathrm{d}x$ in its differential. For each fixed n,

$$\begin{aligned} \mathrm{d}y^n x^{mj-mn+ip} &= n x^{mj-mn+ip} y^{n-1} \mathrm{d}y + (mj - mn + ip) x^{mj-mn+ip-1} y^n \mathrm{d}x \\ &= -mn x^{mj-mn+ip+m-1} y^{n-1} \mathrm{d}x + (mj - mn + ip) x^{mj-mn+ip-1} y^n \mathrm{d}x \end{aligned}$$

If the power of y is r, n could only be $r+1$ or r. If $n = r$, we need to check if $mj-mr+ip-1 = b$ to match the powers of x. Note that if the equality holds, $mj - mr + ip = b \geq 0$, so $y^r x^{mj-mr+ip}$ has nonnegative powers for x automatically. If $n = r + 1$, we need to check if $mj - m(r+1) + ip + m - 1 = b$; in addition, to ensure that $y^r x^{mj-m(r+1)+ip}$ has nonnegative powers for x, we need to check if $mj - m(r+1) + ip \geq 0$. Define indicator functions δ_1, δ_2 as in the statement of Proposition 2.13 for these conditions.
Finally, note that the coefficient of $x^{mj-mn+ip}y^n$ in $(x^i y^j)^p$ is given by $\binom{j}{n}$. Combining them with the coefficients in the differentials and the δ's, we obtain

$$-m\binom{j}{r+1}(r+1)\delta_1 + \binom{j}{r}(mj - mr + ip)\delta_2.$$

□

Ψ has properties similar to Φ's, as in Lemma 2.10 and 2.11.

Lemma 2.14. *Suppose $x^i y^j$, $y^{r_1}x^{b_1}\mathrm{d}x$, $y^{r_2}x^{b_2}\mathrm{d}x$ are valid basis elements for Q or $Q^\vee$ respectively. If the coefficients for $y^{r_1}x^{b_1}\mathrm{d}x$, $y^{r_2}x^{b_2}\mathrm{d}x$ are both nonzero in $\Psi(x^i y^j)$, then $r_1 = r_2$, $b_1 = b_2$.*

Proof. With the same notation in 2.13, note that if $\delta_1 = 1$, $\delta_2 = 1$. Thus if the coefficient is nonzero for $y^r x^b \mathrm{d}x$, $\delta_2 = 1$. Therefore

$$mj - mr_1 + ip - 1 = b_1$$
$$mj - mr_2 + ip - 1 = b_2$$

So $m(r_2 - r_1) = b_1 - b_2 \leq m - 2$, which means $r_1 = r_2$, $b_1 = b_2$. □

Lemma 2.15. *Suppose $x^{i_1}y^{j_1}$, $x^{i_2}y^{j_2}$, $y^r x^b \mathrm{d}x$ are valid basis elements for Q or $Q^\vee$ respectively. If the coefficient for $y^r x^b \mathrm{d}x$ is nonzero in both $\Psi(x^{i_1}y^{j_1})$ and $\Psi(x^{i_2}y^{j_2})$, then $i_1 = i_2$, $j_1 = j_2$.*

Proof. By a similar argument in the proof of Lemma 2.14, $\delta_2 = 1$ and

$$mj_1 - mr + i_1 p - 1 = b$$
$$mj_2 - mr + i_2 p - 1 = b.$$

So $m(j_2 - j_1) = (i_1 - i_2)p$. As $p \nmid m$, $p \mid j_2 - j_1 \leq p - 1$, which means $j_1 = j_2$, $i_1 = i_2$. □

2.4.1. *Worked Example for $p = 5, d = 4$.* We will conclude this section an worked example when $p = 5$ and $f(x) = x^4$. Then

(1) $Q = \langle \frac{y^2}{x}, \frac{y^3}{x}, \frac{y^3}{x^2}, \frac{y^4}{x}, \frac{y^4}{x^2}, \frac{y^4}{x^3} \rangle$;
(2) Φ:
- $\frac{y^3}{x} \mapsto 3 \cdot \frac{y^2}{x}$, $\frac{y^4}{x} \mapsto 4 \cdot \frac{y^3}{x}$
- otherwise 0

(3) Ψ:
- $\ker \Phi = \langle \frac{y^2}{x}, \frac{y^3}{x^2}, \frac{y^4}{x^3}, \frac{y^4}{x^2} \rangle$, $Q^\vee = \langle \mathrm{d}x, x\mathrm{d}x, x^2\mathrm{d}x, y\mathrm{d}x, yx\mathrm{d}x, y^2\mathrm{d}x \rangle$
- $\frac{y^2}{x} \mapsto 3x^2\mathrm{d}x$
- $\frac{y^3}{x^2} \mapsto 2x\mathrm{d}x$
- $\frac{y^4}{x^3} \mapsto \mathrm{d}x$
- $\frac{y^2}{x} \mapsto 3xy\mathrm{d}x$

3. Polarized Dieudonne Module

In this section, we will compute the polarized Dieudonne Module (M, F, V, b) from its Hasse-Witt triple (Q, Φ, Ψ). By [Moo22, Section 2.5], we have

$$M = Q \oplus Q^\vee = H^1(C, \mathcal{O}_C) \oplus H^0(C, \Omega^1_C),$$

so it suffices to compute F, V, b.

3.1 will contain results for both general $f(x) \in k[x]$ and $f(x) = x^m$, while 3.2 and 3.3 compute F and V respectively only for $f(x) = x^m$.

3.1. Polarized Dieudonne Module: b.

The bilinear form b induced by the polarization is explicitly

$$b : M \times M \to k, ((q, \lambda), (q', \lambda')) \mapsto (q, \lambda') - (q', \lambda).$$

where $(-,-) : H^1(C, O_C) \times H^0(C, \Omega^1_C) \to k$ is the pairing induced by Serre duality, explicitly given by

$$((q_p)_{p\in C}, \omega) \mapsto \sum_{p\in C} Res_p(q_p\omega),$$

where Res is defined as in [Har77, Theorem 7.14.1]. Therefore, it suffices to compute $(-,-)$.

We begin with a technical lemma, which is helpful in computing $(-,-)$.

Lemma 3.1. *Let $K(X)$ denote the function field of X. For $0 \le k \le 2p-3$,*

$$\mathrm{Tr}_{K(C)/K(\mathbb{P}^1)}\, y^k = \sum_{0\le t\le p-1} \binom{l_t^k}{t-d_t^k} (f(x))^{l_t^k - t + d_t^k},$$

where $l_t^k = \lfloor \frac{k+t}{p} \rfloor$, $d_t^k = (k+t) \mod p$ for $0 \le t \le p-1$.

Proof. To find the trace, we need to find the coefficient of y^t in y^{k+t} for $0 \le t \le p-1$. Write $k + t = l_t^k p + d_t^k$ where $l_t^k = \lfloor \frac{k+t}{p} \rfloor$, $d_t^k = (k+t) \mod p$.
Note that $l_t^k \le 2$, $d_t^k \le p-1$, and $l_t^k + d_t^k \le p-1$. In particular, the degree of y in $(y + f(x))^{l_t^k - t + d_t^k} y^{d_t^k}$ is less than or equal to $p-1$, so it suffices to extract the coefficient of y^t, which is $\binom{l_t^k}{t-d_t^k}(f(x))^{l_t^k - t + d_t^k}$. Summing up from $t = 0, ..., p-1$, we have the trace being

$$\sum_{0\le t\le p-1} \binom{l_t^k}{t-d_t^k} (f(x))^{l_t^k - t + d_t^k}.$$

□

Theorem 3.2. *Let $q := x^a y^j, \lambda = y^r x^b \mathrm{d}x$ where they are basis elements of $H^1(C, O_C)$ and $H^0(C, \Omega^1_C)$ respectively. Set $i := a + b, k := j + r$. The bilinear pairing $(-,-)$ is given by*

$$-\sum_{0\le t\le p-1} \binom{l_t^k}{t-d_t^k} \sum_{b_0^t,\dots b_m^t} \frac{(l_t^k - t + d_t^k)!}{b_m^t! b_{m-1}^t! \dots b_0^t!} a_{m-1}^{b_{m-1}^t} \dots a_0^{b_0^t}$$

where $l_t^k = \lfloor \frac{k+t}{p} \rfloor$; $d_t^k = (k+t) \mod p$ for $0 \le t \le p-1$; and $(b_0^t, ..., b_m^t)$ goes over all $m+1$ tuples in $\mathbb{Z}^{m+1}_{\ge 0}$ that satisfy $(\star)$;

$$(\star) : \begin{cases} b_m^t + \dots + b_0^t = l_t^k - t + d_t^k \\ m b_m^t + \dots + b_1^t = -1 - i \end{cases}$$

Proof. Note that

$$H^1(C,\mathcal{O}_C)=(\oplus_{p\in C,p\text{ closed}}K(C)/\mathcal{O}_{C,p})/K(C);$$

so in the formula given in 3.1, $[q]\mapsto\begin{cases}q & \text{at }\infty\\ 0 & \text{elsewhere}\end{cases}$, and it suffices to compute $Res_{\tilde\infty}(q\lambda)=Res_{\tilde\infty}(x^{a+b}y^{r+j}\mathrm{d}x)$. Set $i:=a+b, k:=j+r$. By properties of residues and Lemma 3.1,

$$\begin{aligned}Res_{\tilde\infty(x^iy^k\mathrm{d}x)}&=Res_\infty(x^i\operatorname{Tr}_{K(C)/K(\mathbb{P}^1)}y^k\mathrm{d}x)\\&=Res_\infty(x^i\sum_{0\le t\le p-1}\binom{l_t^k}{t-d_t^k}(f(x))^{l_t^k-t+d_t^k}\mathrm{d}x),\end{aligned}$$

where $\tilde\infty\in C$ and $\pi(\tilde\infty)=\infty\in\mathbb{P}^1$, and l_t^k and d_t^k are defined in the statement of Lemma 3.1. Expand the residue linearly and by [Tai14, Theorem 2.5.2], only the terms with x's power being -1 are potentially nonzero. So it suffices to extract the coefficient of x^{-1} in $x^i\sum\limits_{0\le t\le p-1}\binom{l_t^k}{t-d_t^k}(f(x))^{l_t^k-t+d_t^k}$, which is given by

$$N:=\sum_{0\le t\le p-1}\binom{l_t^k}{t-d_t^k}\sum_{b_0^t,\dots b_m^t}\frac{(l_t^k-t+d_t^k)!}{b_m^t!b_{m-1}^t!\dots b_0^t!}a_{m-1}^{b_{m-1}^t}\dots a_0^{b_0^t},$$

with $l_t^k,d_t^k,(b_0^t,\dots,b_m^t)$ defined as in the theorem statement.
Finally, by [Tai14, Theorem 2.5.2], $Res_\infty(\frac{1}{x}\mathrm{d}x)=\operatorname{ord}_\infty(x)$. By linearity of residues,

$$(q,\lambda)=Res_\infty(\frac{N}{x}\mathrm{d}x)=N\operatorname{ord}_\infty(x)=-N.$$

□

We have a simpler expression for the pairing when $f(x)=x^m$ for nonnegative integer m.

Proposition 3.3. *Let $q:=x^iy^j,\lambda=y^rx^b\mathrm{d}x$ where they are basis elements of $H^1(C,O_C)$ and $H^0(C,\Omega_C^1)$ respectively. Set $k:=j+r$, $A:=\frac{k+\frac{1+i}{m}}{p-1}$. When $f(x)=x^m$ for nonnegative integer m, the bilinear pairing $(-,-)$ is given by*

$$(q,\lambda)\mapsto -n\binom{A}{Ap-k}$$

where

$$n=\begin{cases}\min((A+1)p,k+p)-Ap & \text{if } A\in\{0,1,\dots,\lfloor\frac{k+p-1}{p}\rfloor\}\\ 0 & \text{otherwise}\end{cases}$$

Proof. With the same notation and argument in the proof of Theorem 3.2, we need to find the coefficient of x^{-1} in $x^i\sum\limits_{0\le t\le p-1}\binom{l_t^k}{t-d_t^k}(x^m)^{l_t^k-t+d_t^k}$. Note that x's power could only be $m(l_t^k(1-p)+k)$ for all possible values of l_t^k, and the corresponding coefficient is a multiple of $\binom{l_t^k}{l_t^kp-k}$. Set $A:=\frac{k+\frac{1+i}{m}}{p-1}$, so that if $l_t^k=A$, $l_t^k(1-p)+k=-1$; and n to be the number of t's such that $\lfloor\frac{k+t}{p}\rfloor=A$. Then the coefficient of x^{-1} is given by $n\binom{A}{Ap-k}$. With a similar argument in the proof of Theorem 3.2, $(q,\lambda)=-n\binom{A}{Ap-k}$. □

Remark 3.4. The proofs of Theorem 3.2 and Proposition 3.3 are almost identical except that we can find the coefficient of x^{-1} more explicitly in the special case when $f(x)=x^m$.

We can derive an even simpler expression for Proposition 3.3:

Proposition 3.5. *With the assumptions and notations in Proposition 3.3,*

$$(q, \lambda) = \begin{cases} 1 & \text{if } i = -1 \text{ and } k = p-1 \\ 0 & \text{otherwise} \end{cases}$$

Proof. For the pairing to be nontrivial, it is necessary that $\binom{A}{Ap-k}$ is nonzero. In particular, $0 \leq Ap - k \leq A$.

- If $A = 2$: $2p - 2 \leq k \leq 2p$. But $k \leq 2p - 3$. Contradiction.
- If $A = 0$: $0 \leq k \leq 0$. But a direct computations says that the pairing is trivial when $k = 0$. Contradiction.
- If $A = 1$: $p - 1 \leq p$. A direct computation gives 0 when $k = p$; when $k = p - 1$, then $i = -1$. In this case, $n = p - 1$ and $Ap - k = 1$, so $(q, \lambda) = 1$.

□

3.2. **Polarized Dieudonne Module: F.** An explicit description of F is given in [Moo22, Section 2.6]: Let $R_1 = \ker \Phi \subset Q$ and choose some $R_0 \subset Q$ be its complement, F is given by

$$F : M = R_0 \oplus R_1 \oplus Q^\vee \to M = Q \oplus Q^\vee, (r_0, r_1, \lambda) \mapsto (\Phi(r_0), \Psi(r_1))$$

We will assume $f(x) = x^m$. It suffices to give a description of R_0, since we have already computed Φ and Ψ in 2.4. But then by Corollary 2.12,

$$\ker \Phi = \langle x^i y^j | dj - dj' + i \geq 0 \text{ or } \leq -\frac{j'd}{p} \text{ for all } 0 \leq j' \leq j \rangle.$$

Thus we can define R_0 as

$$R_0 \coloneqq \langle x^i y^j | -\frac{j'd}{p} < dj - dj' + ip < 0 \text{ for some } 0 \leq j' \leq j \rangle.$$

3.3. **Polarized Dieudonne Module: V.** By [Moo22, Section 2.6], V is given by

$$V : M = Q \oplus Q^\vee \to M = Q \oplus R_0^\vee \oplus R_1^\vee,$$
$$(q, \lambda) \mapsto (0, \Phi^\vee(\lambda \bmod \operatorname{Im}(\Phi)^\perp), -\Psi^\vee(q \bmod \operatorname{Im}(\Phi))),$$

in which $\Phi^\vee$ and $\Psi^\vee$ are defined by the following property:

$$(\Phi(x), y) = (x, \Phi^\vee(y))^p \text{ for all } x \in Q, y \in Q^\vee$$
$$(\Psi(x), y) = (x, \Psi^\vee(y))^p \text{ for all } x \in Q, y \in Q$$

It suffices to compute $\Phi^\vee$ and $\Psi^\vee$ on a basis, which we will do in 3.3.1 and 3.3.2. We will assume $f(x) = x^m$.

3.3.1. $\Phi^\vee$. We compute $\Phi^\vee$ on a basis of $Q^\vee$. The image of $y^r x^b \mathrm{d}x$ is given by the following procedure:

(1) Find the unique $x^i y^j$ that pairs nontrivially with $y^r x^b \mathrm{d}x$ (i.e. $i = -1-b$, $j = p-1-r$).
(2) Look for the unique (i_2, j_2) such that $\Phi(x^{i_2} y^{j_2}) \in span(x^i y^j)$.
(3) $\begin{cases} y^r x^b \mathrm{d}x \mapsto \binom{j_2}{j} y^{p-1-j_2} x^{-1-i_2} \mathrm{d}x & \text{if such } (i_2, j_2) \text{ exists} \\ 0 & \text{otherwise} \end{cases}$

We will justify this algorithm along with the algorithm for $\Psi^\vee$ with an example by the end of 3.3.2.

3.3.2. $\Psi^\vee$. We compute $\Psi^\vee$ on a basis of Q. The image of x^iy^j is given by the following procedure:

- Find the unique x^iy^j that pairs nontrivially with $y^rx^b\mathrm{d}x$ (i.e. $b=-1-i$, $r=p-1-j$).
- Look for the unique (i_2,j_2) such that $\Psi(x^{i_2}y^{j_2})\in span(y^rx^b\mathrm{d}x)$.
- $\begin{cases} y^rx^b\mathrm{d}x \mapsto Ay^{p-1-j_2}x^{-1-i_2}\mathrm{d}x & \text{if such } (i_2,j_2) \text{ exists} \\ 0 & \text{otherwise} \end{cases}$ where A is given in Proposition 2.13 (with i_2, j_2 replacing i,j in that formula).

It suffices to illustrate the correctness of the algorithms in 3.3.1 and 3.3.2 with an example. The general case works similarly.

Example 3.6. Let $p=5$ and $f(x)=x^4$. Suppose we want to compute $\Psi^\vee(\frac{y^3}{x^2})$. By the defining property, for any $q\in Q$,

$$(q,\Psi^\vee(\frac{y^3}{x^2}))^5=(\frac{y^3}{x^2},\Psi(q)).$$

Note that $xy\mathrm{d}x$ is the unique basis element that pairs nontrivially with $\frac{y^3}{x^2}$.

- If $\Psi(q)\in span(xy\mathrm{d}x)$, then the right hand side is nonzero (as $(\frac{y^3}{x^2},xy\mathrm{d}x)=1$). In particular, $\Psi(\frac{y^4}{x^2})=3xy\mathrm{d}x$, so
$$(\frac{y^4}{x^2},\Psi^\vee(\frac{y^3}{x^2}))^5=(\frac{y^3}{x^2},\Psi(\frac{y^4}{x^2}))=3.$$
(We can always go back to at most one basis element like this, by Lemma 2.14 and 2.15.) $\frac{y^4}{x^2}$ only pairs nontrivially with $3x\mathrm{d}x$. Therefore, the coefficient of $x\mathrm{d}x$ in $\Psi^\vee(\frac{y^3}{x^2})$ is 3.
- Otherwise, by Lemma 2.14, $\Psi(q)\in span(xy\mathrm{d}x)^\perp$. Then the right hand side is always 0, and the unique $y^rx^b\mathrm{d}x$ that pairs with each basis element other than $\frac{y^4}{x^2}$ always has coefficient zero in the image of $\Psi^\vee(\frac{y^3}{x^2})$. This means that
$$\Psi^\vee(\frac{y^3}{x^2})=3x\mathrm{d}x.$$

Remark 3.7. (1) Suppose we want to find $\Psi^\vee(v)$, and we don't have such a q such that $\Psi(q)$ is in the span of the unique element that pairs with v nontrivially, then $\Psi^\vee(v)=0$ by the second case in Example 3.6.

(2) If we replace Ψ with Φ everywhere and the corresponding lemmas from Lemma 2.14 and 2.15 to Lemma 2.10 and 2.11, every argument in Example 3.6 still holds for $\Phi^\vee$, which justifies the procedure given in Section 3.3.1.

A python implementation of 3.3.1 and 3.3.2 is given in `EOType_AScurves.py` and explained in the Appendix.

3.3.3. *Worked Example for $\Phi^\vee,\Psi^\vee$: $p=5, d=4$.* We return to the example when $p=5$ and $f(x)=x^4$ in Section 2.4.1 and Example 3.6. Then $\Phi^\vee$ and $\Psi^\vee$ are given by:

(1) $\Phi^\vee$:
  - $\mathrm{d}x, x\mathrm{d}x, x^2\mathrm{d}x, xy\mathrm{d}x\mapsto 0$
  - $y^2\mathrm{d}x\mapsto 3y\mathrm{d}x$
  - $y\mathrm{d}x\mapsto -\mathrm{d}x$

(2) $\Psi^\vee$:
- $\frac{y^3}{x}, \frac{y^2}{x} \mapsto 0$
- $\frac{y^3}{x^2} \mapsto 3x\mathrm{d}x$
- $\frac{y^4}{x^2} \mapsto 2xy\mathrm{d}x$
- $\frac{y^4}{x^3} \mapsto 3y^2\mathrm{d}x$
- $\frac{y^4}{x} \mapsto x^2\mathrm{d}x$

4. Final Type

We will compute the final type representation for the Ekedahl-Oort type from a polarized Dieudonne Module (M, F, V, b) in this section. The advantage of the final type is that it uniquely determines a polarized Dieudonne module and it is canonical in the sense that it doesn't depend on a chosen basis. We will describe how it is constructed from a polarized Dieudonne Module in 4.1, and provide two algorithms that will combine to compute the final type for the special case when $f(x) = x^m$.

4.1. **The Construction of the Final Type.** Given a polarized Dieudonne module (M, F, V, b), we aim to compute its final type, which is given in the following description: Set $N := V^{-1}(0) = F(M)$, and we have $0 \subseteq N \subseteq M$. We can go left to N by taking $F(N)$ and go right by taking $V^{-1}(N)$ in terms of inclusion. For each vector space we obtain in process, we will take F and V^{-1} in the same fashion and we can obtain such a sequence called the canonical flag, once it stabilizes:

$$0 = N_1 \subseteq ... \subseteq N_t = M. \tag{4.1}$$

We then fill in the missing dimensions with vector spaces in between; the final type remain well defined when they are chosen arbitrarily, as long as they respect inclusion. Relabel $N_1, ...N_t$ in 4.1, we would have the full flag given by

$$0 = N'_1 \subseteq ... \subseteq N'_{2g} = M. \tag{4.2}$$

Set $V_i = \dim F(N'_i)$. The Final type is given by

$$[V_1, ...V_g]. \tag{4.3}$$

4.2. **Computing Final Type from Polarized Dieudonne Modules.**

4.2.1. *The Canonical Flag.* We start by computing the Canonical Flag in 4.1. The algorithm follows from the description in 4.1 and is given in Algorithm 1; the code is given in `EOType_AScurves.py`. Be aware that the lists in Algorithm 1 are indexed from 1 instead of 0.

4.2.2. *The Final Type.* We will then compute the final type in 4.3. The algorithm is given in Algorithm 2 and the code is given in `EOType_AScurves.py`. Be aware that the lists in Algorithm 2 are indexed from 1 instead of 0.

Remark 4.4. We conclude the section with some remarks on Algorithm 2:

(1) F sends every basis element from Q uniquely to another element in $Q \oplus Q^\vee$, and ignores everything from $Q^\vee$. Therefore, line 12, 18 sets $V[i]$ to be $\dim F(A)$.

Algorithm 1 The Canonical Flag

Input: p, m

Output: N, the canonical flag given by a list indexed by dimension

```
N ← [None, None, ..., None]          ▷ W is a list of length 2×genus initialized by None
Func FlagHelper(W)                   ▷ W is a vector space
    if dim W = 0 then
        End Recursion
    else if N[dim W] is not None then
        End Recursion
    end if
    N[dim W] ← W
    FlagHelper(F(W))
    FlagHelper(V^{-1}(W))
end Func
W ← V^{-1}(0)
FlagHelper(W)
```

Algorithm 2 The Final Type

Input: p, m

Output: $[V[1], V[2], ..., V[g]]$

```
CFlag ← Canonical Flag computed by Algorithm 2
g ← (p−1)(m−1)/2, the genus of the curve
V ← [0, 0, ...0]                     ▷ empty list with length being 2g
i ← 1
while i ≤ 2g do
    A ← CFlag[i]
    t ← The next nonempty dimension in C, and −1 if it doesn't exist
    if t = −1 then
        break
    else if |basis(A) ∩ basis(Q^∨)| < |basis(CFlag[t]) ∩ basis(Q^∨)| then
        while i < t do
            V[i] ← |basis(A) ∩ basis(Q)|
            i ← i + 1
        end while
    else
        counter ← 0
        while i < t do
            V[i] = |basis(A) ∩ basis(Q)|+ counter
            i ← i + 1
            counter ← counter + 1
        end while
    end if
end while
```

(2) Suppose N_a, N_{a+1} in the canonical flag have $\dim N_{a+1} - \dim N_a \geq 2$. If there exist $u \in Q \cap (N_{a+1} \backslash N_a)$, $v \in Q^\vee \cap (N_{a+1} \backslash N_a)$, then the final type would depend on the order of adjoining u or v. Therefore, $N_{a+1} \backslash N_a \subset Q^\vee$ or $N_{a+1} \backslash N_a \subset Q$. In the first case, $\dim F$ remains the same every time we adjoin an element from Q; in the second case, $\dim F$ increases by 1 every time we adjoin an element from $Q^\vee$. These conditions are expressed in line 10 and 15 respectively in Algorithm 2.

5. Directions for Future Work

We will conclude the report with a two directions for future work on the subject matter.

(1) We wish to obtain an easy formula for the final type, at least for $f(x) = x^m$ for non-negative integer m. With the help of the python program described in the Appendix, we conjecture the following:

Conjecture 5.1. *When $f(x) = x^m \in k[x]$ and $m \mid p-1$, the final type starts with $\frac{p-1}{m}$ zeros.*

We wonder if there are more patterns like this in the final type and if we can prove them as a formula in general.

(2) Compute the Hasse Witt Triple and Polarized Dieudonne Module for general $f \in k(x)$. [EP10, Section 4] has computed the first cohomology for general $f \in k(x)$ and a formula for Φ follows easily from their work. However, the challenge lies in obtaining an explicit description of Ψ – the affine cover for computing the first cohomology in that case requires more affine opens, and the description we have in Section 2.3.1 does not apply.

Appendix: Python Implementation

A. **Overview.** We have implemented our method with Python 3.9.6 to compute the Ekedahl-Oort type of Artin-Schreier curves when they are defined by $y^p - y = f(x)$ where $f(x) = x^m$ for non-negative integer m. The source code can be accessed on `https://nancium.notion.site/Research-fe3fc9d318ad44f09ead6305305bee85?pvs=4` in "Source Code" under the drop down list "Ekedahl-Oort Types of Artin-Schreier Curves". The file name is `EOType_AScurves.py`.

To run the code, please make sure that the python installation is 3.8 or later. The reader can run `EOType_AScurves.py` in any environment that supports python (i.e. IDLE) and send commands through the shell. We have defined some Hasse-Witt Triples and Polarized Dieudonne Modules in the source code (after `if __name__ == "__main__":`) section, so the reader can easily experiment with those objects.

B. **The class `HasseWittTriple`.**

B.1. *Initialization.* A `HasseWittTriple` object is initialized with two parameters: p and m, where $p = char(k)$ and $m = \deg f$. It will raise a value error if $p \mid m$. For the rest of this section, suppose we have initialized an HasseWittTriple object `HWT` with p and m:

B.2. ***HWT.H1_basis()***. Return a basis of Q in Theorem 2.2 as a set of tuples. Each tuple is in the form of (i, j), and this means $x^i y^j$ is a basis element of Q. An exception is raised with an error message when $p \nmid m$.

B.3. ***HWT.Phi(display = False)***. Return Φ on a basis as a dictionary. The key-value pairs are the form of $(i, j) : (a, (i', j'))$ where a is nonzero, which means $\Phi(x^i y^j) = a x^{i'} y^{j'}$. If `display = True`, it will print the key-value pairs line by line.

B.4. ***HWT.ker_Phi()***. Return the kernel of Φ as a set of tuples in the form of (i, j), which means $x^i y^j$ is a basis element of the kernel of Φ.

B.5. ***HWT.valid_diff_basis()***. Return a basis of $Q^\vee = H^0(C, \Omega^1_C)$ given in 2.3.1 as a set of tuples. Each tuple is in the form of (r, b), which means $y^r x^b \mathrm{d}x$ is a basis element in this basis of $Q^\vee$.

B.6. ***HWT.Psi(display = False)***. Return the image of Ψ on a basis as a dictionary. The key-value pairs are the form of $(i, j) : (a, (r, b))$ where a is nonzero, which means $\Psi(x^i y^j) = a y^r x^b \mathrm{d}x$. If `display = True`, it will print the key-value pairs line by line.

C. **The class `DieudonneModule`.**

C.1. *Initialization.* A `DieudonneModule` object is initialized with two parameters: p and m, where $p = char(k)$ and $m = \deg f$. Upon initialization, it will initialize a `HasseWittTriple` object from p, m and automatically find the basis for Q and $Q^\vee$. For the rest of this section, suppose we have initialized an `DieudonneModule` object `DM` with p and m:

C.2. ***DM.pairing(q_i, q_j, lbd_r,lbd_b)***. Given an input representing $q = x^{q\_i} y^{q\_j}$ and $\lambda = y^{lbd\_r} x^{lbd\_b} \mathrm{d}x$, return (q, λ).

C.3. ***DM.b_bilinear(q1, lbd1, q2, lbd2)***. Given an input of tuples representing $q = x^{q[0]} y^{q[1]}$ and $lbd = y^{lbd[0]} x^{lbd[1]} \mathrm{d}x$, return $b((q1, lbd1), (q2, lbd2))$.

C.4. ***DM.Phi_dual()***. Return the image of $\Phi^\vee$ on a basis as a dictionary. The key-value pairs are the form of $(r, b) : (a, (r', b'))$ where a is nonzero, which means $\Phi^\vee(y^r x^b \mathrm{d}x) = a y^{r'} x^{b'} \mathrm{d}x$.

C.5. *`DM.Psi_dual()`.* Return the image of $\Psi^\vee$ on a basis as a dictionary. The key-value pairs are the form of $(i, j) : (a, (r, b))$ where a is nonzero, which means $\Psi^\vee(x^i y^j) = a y^r x^b \mathrm{d}x$.

D. The class `Final Type`.

D.1. *Initialization.* A `FinalType` object is initialized with two parameters: p and m, where $p = char(k)$ and $m = \deg f$. Upon initialization, it will initialize a `DieudonneModule` object from p, m and computes the kernel of V. For the rest of this section, suppose we have initialized an `FinalType` object `FT` with p and m:

D.2. *`FT.kerV()`.* Return the kernel of V on a basis, given by a list containing two sets in the form of $\{(i_1, j_1), (i_2, j_2), ...\}$ (denoting $x^i y^j$) and $\{(r_1, b_1), (r_2, b_2), ...\}$ (denoting $y^r x^b \mathrm{d}x$).

D.3. *`FT.F(lst)`.* Given a vector space on a basis as a list containing two sets in the form of $\{(i_1, j_1), (i_2, j_2), ...\}$ (denoting $x^i y^j$) and $\{(r_1, b_1), (r_2, b_2), ...\}$ (denoting $y^r x^b \mathrm{d}x$), return the image of F on a basis, given by a list in the same format.

D.4. *`FT.V_preim(lst)`.* Given a vector space on a basis as a list containing two sets in the form of $\{(i_1, j_1), (i_2, j_2), ...\}$ (denoting $x^i y^j$) and $\{(r_1, b_1), (r_2, b_2), ...\}$ (denoting $y^r x^b \mathrm{d}x$), return the preimage of V on a basis, given by a list in the same format.

D.5. *`FT.FT_Tree(display = False)`.* Return nothing, save the computed canonical tree in a list, where the $i - th$ element in the list saves the corresponding $i + 1$-dimensional space on a basis suppose it exists, and is $None$ if it doesn't. If `display = True`, print each dimension and a basis of the corresponding vector space line by line.

D.6. *`FT.Final_type()`.* Return the final type given by a list $[N_1, N_2, ..., N_g]$, where g is the genus of the curve. It will run `FT.FT_Tree()` automatically, so there is no need to call `FT.FT_Tree()` before calling this function.

Acknowledgments

My sincerest thanks go to Dr. Patrick Daniels for mentoring this project; his encouragement and support helped our project proceed smoothly, and moreover piqued my interest in arithmetic geometry. I'm also grateful to Professor Michael Zieve for briefly mentioning Silverman's AEC when I was taking MATH 494 (WN 2022), when I first learned about the field of arithmetic geometry, and this eventually led to me working with Dr. Daniels on such a project.

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